Lecture 13: Cooperative Game Theory Coalition Game

Yi, Yung (이융)
KAIST, Electrical Engineering
http://lanada.kaist.ac.kr
yiyung@kaist.edu

Introduction

- Introduction to cooperative game
- Bargaining solution
- Coalitional game

Backgrounds

- "Cooperative" may be misleading
  - Does not mean that each agent is agreeable and follow arbitrary instr uctions
  - Means that the basic modeling unit is the group rather than the indivi dual agent
- Concepts
  - What groups of agents can achieve
  - Not concerned with
    - How the agents make individual choices within a coalition
    - How they coordinate

Coalitional Games: Preliminaries

- **Definition** of a coalitional game (*N,v*)
  - A set of players ***N***, ***a coalition*** S is a group of cooperating players (subset of N)
  - Worth (utility) of a coalition ***v***
    - In general, **payoff v(S)** is a real number that represents the gain resulting from a coalition S in the game (N,v)
    - **v(N)** is the worth of forming the coalition of all users, known as the **grand coalition**
  - User payoff x_i : the portion of ***v(S)*** received by a player ***i*** in coalition S

Coalitional Games: Utility

- Transferable utility (TU)
  - The worth ***v(S)*** of a coalition *S* can be distributed arbitrarily among the players in a coalition hence,
  - ***v(S)*** is a **function** from the power set of N over the real line
- Non-transferable utility (NTU)
  - The payoff that a user receives in a coalition is pre-determined, and hence the value of a coalition cannot be described by a function
  - ***v(S)*** is a set of payoff vectors that the players in *S* can achieve

$$v(S) \subseteq \mathbb{R}^{|S|}$$

Definition

- Transferable utility assumption:
  - the payoffs to a coalition may be freely redistributed among its members.
  - satisfied whenever there is a universal currency that is used for exchange in the system
  - means that each coalition can be assigned a single value as its payoff.

Definition (Coalitional game with transferable utility)

A coalitional game with transferable utility is a pair (N, v), where

- N is a finite set of players, indexed by i; and
- $v : 2^N \mapsto \mathbb{R}$ associates with each coalition $S \subseteq N$ a real-valued payoff $v(S)$ that the coalition's members can distribute among themselves. We assume that $v(\emptyset) = 0$.

Using Coalitional Game Theory

Questions we use coalitional game theory to answer:

1. Which coalition will form?
2. How should that coalition divide its payoff among its members?

The answer to (1) is often "the grand coalition"—the name given to the coalition of all the agents in N—though this can depend on having made the right choice about (2).

Voting Game

Our first example considers a social choice setting.

Example (Voting game)

The parliament of Micronesia is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

More generally, in a voting game, there is a set of agents N and a set of coalitions $\mathcal{W} \subseteq 2^N$ that are winning coalitions, that is, coalitions that are sufficient for the passage of the bill if all its members choose to do so. To each coalition $S \in \mathcal{W}$, we assign $v(S) = 1$, and to the others we assign $v(S) = 0$.

Airport Game

Our second example concerns sharing the cost of a public good, along the lines of the road-building referendum.

Example (Airport game)

A number of cities need airport capacity. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.
This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Minimum Spanning Tree

Next, consider a situation in which agents need to get connected to the public good in order to enjoy its benefit. One such setting is the problem of multicast cost sharing.

Example (Minimum spanning tree game)

A group of customers must be connected to a critical service provided by some central facility, such as a power plant or an emergency switchboard. In order to be served, a customer must either be directly connected to the facility or be connected to some other connected customer. Let us model the customers and the facility as nodes on a graph, and the possible connections as edges with associated costs.
This situation can be modeled as a coalitional game (N, v). N is the set of customers, and $v(S)$ is the cost of connecting all customers in S directly to the facility minus the cost of the minimum spanning tree that spans both the customers in S and the facility.

Auction

Finally, consider an efficient auction mechanism. Our previous analysis treated the set of participating agents as given. We might instead want to determine if the seller would prefer to exclude some interested agents to obtain higher payments. To find out, we can model the auction as a coalitional game.

Example (Auction game)

Let N_B be the set of bidders, and let 0 be the seller. The agents in the coalitional game are $N = N_B \cup \{0\}$. Choosing a coalition means running the auction with the appropriate set of agents. The value of a coalition S is the sum of agents' utilities for the efficient allocation when the set of participating agents is restricted to S. A coalition that does not include the seller has value 0, because in this case a trade cannot occur.

Superadditive games

Definition (Superadditive game)

A game $G = (N, v)$ is superadditive if for all $S, T \subset N$, if $S \cap T = \emptyset$, then $v(S \cup T) \geq v(S) + v(T)$.

- Superadditivity is justified when coalitions can always work without interfering with one another
  - the value of two coalitions will be no less than the sum of their individual values.
  - implies that the grand coalition has the highest payoff
- All our examples are superadditive.

Convex games

An important subclass of superadditive games are the convex games.

Definition (Convex game)

A game $G = (N, v)$ is convex if for all $S, T \subset N$,
$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

- Convexity is a stronger condition than superadditivity.
  - However, convex games are not too rare in practice.
  - E.g., the airport game is convex.
- Convex games have a number of useful properties, as we will see later.

Analyzing coalitional games

1. Which coalition will form?
   - we'll consider cases where the answer is the grand coalition
   - makes sense for superadditive games, like all our examples
2. How should the coalition divide its payoff?
   - in order to be fair
   - in order to be stable

Terminology:

- $\psi : \mathbb{N} \times \mathbb{R}^{2^{|N|}} \mapsto \mathbb{R}^{|N|}$: payoff division given a game
  - $\psi(N, v)$ is a vector of payoffs to each agent, explaining how they divide the payoff of the grand coalition
  - $\psi_i(N, v)$ is i's payoff
- shorthand: $x \in \mathbb{R}^N$: payoffs to each agent in N, when the game is implicit.

Axiomatizing fairness: Symmetry

As we did in social choice, let us describe fairness through axioms.

- i and j are interchangeable if they always contribute the same amount to every coalition of the other agents.
  - for all S that contains neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.
- The symmetry axiom states that such agents should receive the same payments.

Axiom (Symmetry)

For any v, if i and j are interchangeable then $\psi_i(N, v) = \psi_j(N, v)$.

Axiomatizing fairness: Dummy Player

- i is a dummy player if the amount that i contributes to any coalition is exactly the amount that i is able to achieve alone.
  - for all S such that $i \notin S$, $v(S \cup \{i\}) - v(S) = v(\{i\})$.
- The dummy player axiom states that dummy players should receive a payment equal to exactly the amount that they achieve on their own.

Axiom (Dummy player)

For any v, if i is a dummy player then $\psi_i(N, v) = v(\{i\})$.

Axiomatizing fairness: Additivity

- Consider two different coalitional game theory problems, defined by two different characteristic functions v_1 and v_2, involving the same set of agents.
- The additivity axiom states that if we re-model the setting as a single game in which each coalition S achieves a payoff of $v_1(S) + v_2(S)$, the agents' payments in each coalition should be the sum of the payments they would have achieved for that coalition under the two separate games.

Axiom (Additivity)

For any two v_1 and v_2, we have for any player i that $\psi_i(N, v_1 + v_2) = \psi_i(N, v_1) + \psi_i(N, v_2)$, where the game $(N, v_1 + v_2)$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for every coalition S.

Shapley Value

Theorem

Given a coalitional game (N, v), there is a unique payoff division $x(v) = \phi(N, v)$ that divides the full payoff of the grand coalition and that satisfies the Symmetry, Dummy player and Additivity axioms.

What is this payoff division $\phi(N, v)$? It is called the Shapley value.

Definition (Shapley value)

Given a coalitional game (N, v), the Shapley value of player i is given by

$$\phi_i(N, v) = \frac{1}{N!} \sum_{S \subseteq N \setminus \{i\}} |S|!(|N| - |S| - 1)! \Big[v(S \cup \{i\}) - v(S) \Big].$$

Understanding the Shapley Value

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This captures the "average marginal contribution" of agent i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

- imagine that the coalition is assembled by starting with the empty set and adding one agent at a time, with the agent to be added chosen uniformly at random.

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- Within any such sequence of additions, look at agent i's marginal contribution at the time he is added.

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This captures the "average marginal contribution" of agent i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

- If he is added to the set S, his contribution is $[v(S \cup \{i\}) - v(S)]$.

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- Now multiply this quantity by the $|S|!$ different ways the set S could have been formed prior to agent i's addition and by the $(|N| - |S| - 1)!$ different ways the remaining agents could be added afterward.

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- Finally, sum over all possible sets S and obtain an average by dividing by $|N|!$, the number of possible orderings of all the agents.

Shapley Value Example

Consider the Voting game:

- A, B, C, and D have 45, 25, 15, and 15 votes
- 51 votes are required to pass the $100 million bill
- A is in all winning coalitions, but doesn't win alone
- B, C, D are interchangeable: they always provide the same marginal benefit to each coalition
  - they add $100 million to the coalitions $\{B, C\}$, $\{C, D\}$, $\{B, D\}$ that do not include them already and to $\{A\}$
  - they add $0 to all other coalitions
- Grinding through the Shapley value calculation (see the book), we get the payoff division $(50, 16.66, 16.66, 16.66)$, which adds up to the entire $100 million.

Stable payoff division

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members.
  - However, this analysis ignored questions of stability.
- Would the agents be willing to form the grand coalition given the way it will divide payments, or would some of them prefer to form smaller coalitions?
  - Unfortunately, sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.
  - Considering the majority voting example, while A does not have a unilateral motivation to vote for a different split, A and B have incentive to defect and divide the $\$100$ million between them (e.g., $(75, 25)$).

The Core

- Under what payment divisions would the agents want to form the grand coalition?
- They would want to do so if and only if the payment profile is drawn from a set called the core.

Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) if and only if

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

- The sum of payoffs to the agents in any subcoalition S is at least as large as the amount that these agents could earn by forming a coalition on their own.
- Analogue to Nash equilibrium, except that it allows deviations by groups of agents.

Existence and Uniqueness

1. Is the core always nonempty?
2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty? No.
   - Consider again the voting game.
   - The set of minimal coalitions that meet the required 51 votes is $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$.
   - If the sum of the payoffs to parties B, C, and D is less than \$100 million, then this set of agents has incentive to deviate.
   - If B, C, and D get the entire payoff of \$100 million, then A will receive \$0 and will have incentive to form a coalition with whichever of B, C, and D obtained the smallest payoff.
   - Thus, the core is empty for this game.
2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty?
2. Is the core always unique? No.
   - Consider changing the example so that an 80% majority is required
   - The minimal winning coalitions are now $\{A, B, C\}$ and $\{A, B, D\}$.
   - Any complete distribution of the \$100 million among A and B now belongs to the core
     - all winning coalitions must have both the support of these two parties.

Positive results

We say that a player i is a veto player if $v(N \setminus \{i\}) = 0$.

Theorem

In a simple game the core is empty iff there is no veto player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0.

Theorem

Every convex game has a nonempty core.

Theorem

In every convex game, the Shapley value is in the core.

Summary