Lecture 12: Cooperative Game Theory Nash Bargaining Solution

Yi, Yung (이융)
KAIST, Electrical Engineering
http://lanada.kaist.ac.kr
yiyung@kaist.edu

Where We Are?

- Non-cooperative game theory
  - Various games
  - Normal-form game
  - Extensive-form game
  - Repeated game
  - Bayesian game
  - …
- Cooperative game theory
  - Bargaining (today)
  - Coalitional game

Introduction

- Introduction to cooperative game
- Bargaining solution
  - Nash Bargaining Solution
  - Kalai – Smorodinsky Bargaining Solution
  - Rubinstein Bargaining Process
  - Examples
- Coalitional game

Cooperative Game Theory

- Underlying situation
  - Players have mutual benefit to cooperate

- Two categories
  - Bargaining problems
  - Coalitional game

Introduction to Bargaining

- Bargaining situation
  - A number of individuals have a **common** interest to cooperate (e.g., trade or sharing of resource), but a conflicting interest on **how to cooperate (terms of agreement, 계약조건)**
- Key tradeoff
  - Players wish to reach an agreement rather than disagree.
  - But, each player is self-interested

Introduction to Bargaining

- What is bargaining?
  - Process through which the players on their own attempt to reach an agreement
  - Can be tedious, involving offers and counter-offers, negotiations, etc.

- Bargaining theory studies these situations, their outcome, and the bargaining process

Examples

- Painting
  - Seller: values at 1000$
  - Buyer: values at 1500$
  - What's the final price?

- Sharing Resource
  - Shared resource among nodes
  - How to share?
  - Every node wants to get larger portion of resource

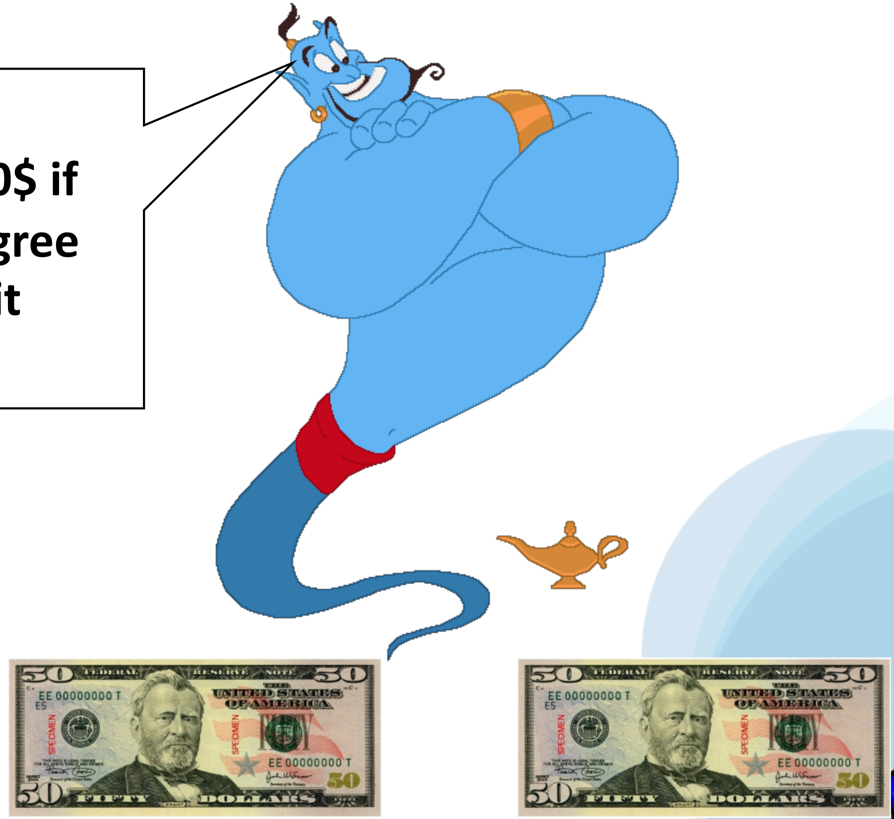
Introduction

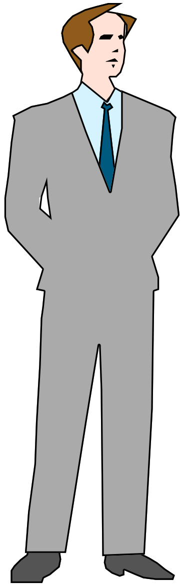
- Key issues in bargaining
  1. The players must inspect efficiency and the effect of delay and disagreement on it, i.e., jointly efficient
     - They seek a jointly efficient mutual agreement
  2. Distribution of the gains from the agreement
     - Which point from the efficient set must the players select?
  3. What are the joint strategies that the players must choose to get the desired outcome?
  4. How to finally enforce the agreement?

- Link to game theory
  - Issues 1 and 2 are tackled traditionally by cooperative game theory
  - Issues 3 and 4 are strongly linked to non-cooperative game theory

Motivating Example (1)

I can give you 100$ if and only if you agree on how to share it





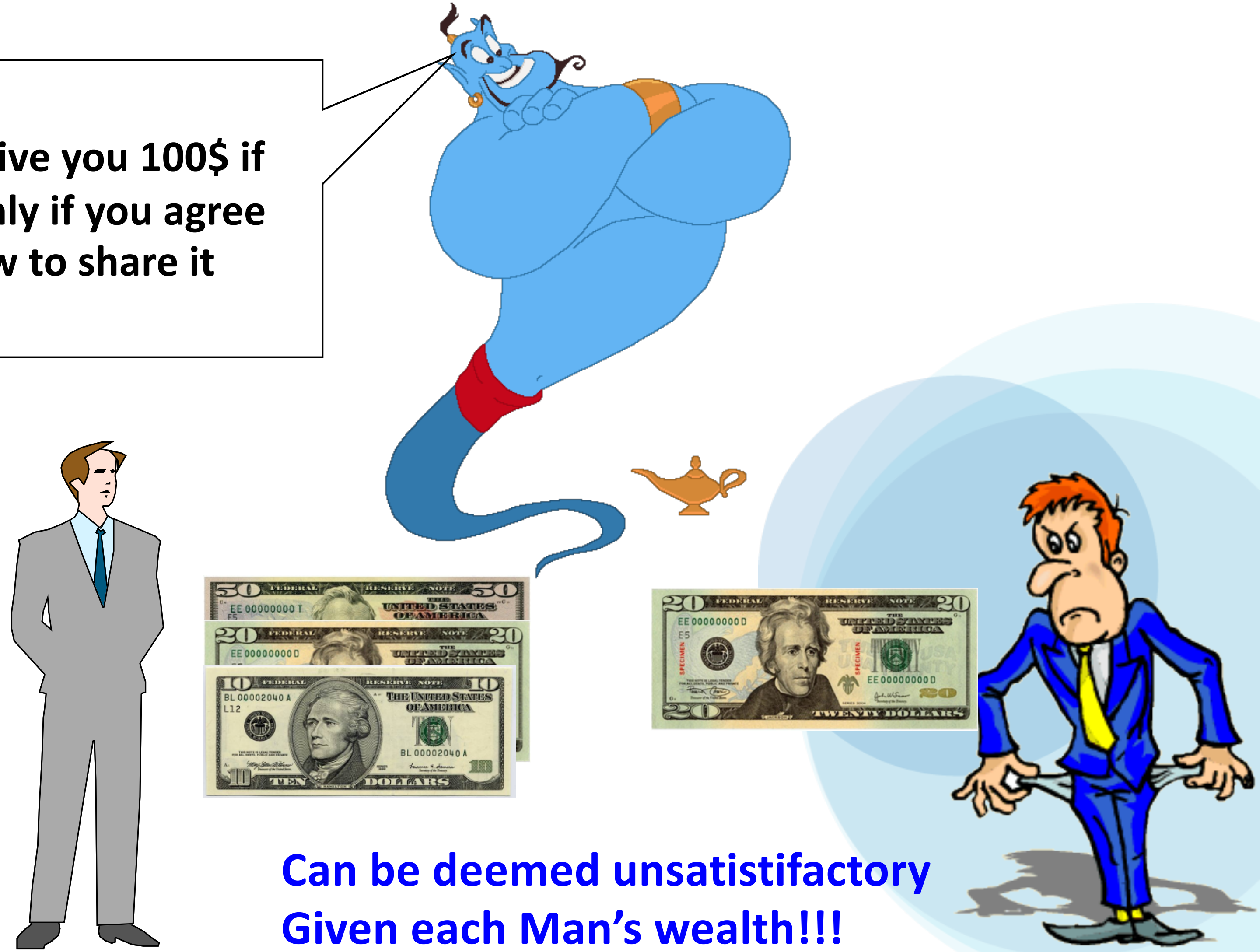
Can be deemed unsatistifactory
Given each Man's wealth!!!

Rich Man (Wealth = $\$10^{10}$)

Poor Man (Wealth = $\$10$)

Motivating Example (2)

I can give you 100$ if and only if you agree on how to share it



Can be deemed unsatistifactory
Given each Man's wealth!!!

Rich Man (Wealth = $\$10^{10}$)

Poor Man (Wealth = \$10)

The Nash Bargaining Solution

- **John Nash**'s approach
  - When presented with a bargaining problem such as the rich man – poor man case, how can we pick a reasonable outcome?
  - Interested in the **outcome** rather than the **process**

- **The Nash Bargaining Solution** was proposed in 1950 using an axiomatic approach and is considered as one of the key foundations of bargaining problems

Setup

- Consider two players for simplicity (i=1,2)
- Outcome space $\mathcal{X} \cup \{D\}$
  - *D*: the outcome of disagreement
- Utility function u_i on $\mathcal{S}$

$$\mathcal{S} = \{(u_1(x_1), u_2(x_2)) \mid x = (x_1, x_2) \in \mathcal{X}\}$$

- Utility of disagreement $d = (d_1, d_2)$, where $d_i = u_i(D)$
- A Nash bargaining problem is defined by the pair *(S,d)*

The Nash Bargaining Solution

- Can we find a *bargaining solution*, i.e., a function f that specifies a **unique** outcome $f(S,d) \in S$?

- Axiomatic approach proposed by Nash
  - Axiom 1: Feasibility
  - Axiom 2: Pareto efficiency
  - Axiom 3: Symmetry
  - Axiom 4: Invariance to linear transformation
  - Axiom 5: Independence of irrelevant alternatives

The Nash Bargaining Solution

- Axiom 1: Feasibility

- Feasibility implies that
  - The outcome of the bargaining process, denoted (u^*,v^*) cannot be worse than the disagreement point $d = (d_1,d_2)$, i.e., $(u^*,v^*) \geq (d_1,d_2)$
  - Strict inequality is sometimes defined

- Trivial requirement but important: the disagreement point is a benchmark and its selection is very important in a problem!

The Nash Bargaining Solution

- Axiom 2: Pareto efficiency
  - Players need to do as well as they can without hurting one another

- At the bargaining outcome, no player can improve without decreasing the other player's utility
  - Pareto boundary of the utility region

- Formally, no point (u,v) ϵ S exists such that $u > u^*$ **and** $v \geq v^*$ or $u \geq u^*$ **and** $v > v^*$

The Nash Bargaining Solution

- Axiom 3: Symmetry
  - If the utility region is symmetric around a line with slope 45 degrees then the outcome will lie on the line of symmetry
  - Formally, if $d_1 = d_2$ and S is symmetric around $u = v$, then $u^* = v^*$

- Axiom 4: Invariance to linear transformation
  - The bargaining outcome varies linearly if the utilities are scaled using an affine transformation

The Nash Bargaining Solution

- Axiom 5: Independence of irrelevant alternatives
  - If the solution of the bargaining problem lies in a subset *U* of *S*, then the outcome does not vary if bargaining is performed on *U* instead of *S*
  - *If the solution of a larger set is a member of a smaller set, then this solution is also the solution of the smaller set*

The Nash Bargaining Solution

- Theorem. Nash showed that there exists a unique solution f satisfying the axioms, and it takes the following form:

$$(u^*, v^*) = f(S, d) = \max_{(u,v)\in S} (u - d_1)(v - d_2)$$

When $d_1 = d_2 = 0$, this is equivalent to the famous solution of telecommunication networks: Proportional fairness

Known as the Nash product

Rich man – poor man problem revisited

- Considering logarithmic utilities and considering that what the men’s current wealth is as the disagreement point
  - The Nash solution dictates that the rich man receives **a larger** share of the 100$

- Is it fair?
  - Fairness is subjective here, the rich man has more bargaining power so he can threaten more to stop the deal
    - The poor man also values little money big as he is already poor!
  - Variant of the problem considers the 100$ as a debt, and, in that case, the NBS becomes fair, the richer you are the more you pay!

The Nash Bargaining Solution

- The NBS is easily extended to the N-person case
  - The utility space becomes N-dimensional and the disagreement point as well
  - Computational complexity definitely increases and coordination on a larger scale is required
- Solution to the following maximization problem

$$(u_1^*, \ldots, u_N^*) = f(S, d) = \max_{(u_1, \ldots, u_N) \in S} \prod_{i=1}^{N} (u_i - d_i)$$

The Nash Bargaining Solution

- If we drop the Symmetry axiom we define the Generalize Nash Bargaining Solution
- Solution to the following maximization problem

$$(u_1^*, \ldots, u_N^*) = f(S, d) = \max_{(u_1, \ldots, u_N) \in S} \prod_{i=1}^{N} (u_i - d_i)^{\alpha_i}$$

Value between 0 and 1 representing the bargaining power of player *i*
If equal bargaining powers are used, this is equivalent to the NBS

Nash Bargaining Solution – Summary

- The NBS/GNBS are a very interesting concept for allocating utilities in a bargaining problem
  - Provide Pareto optimality
  - Account for the bargaining power of the players but..
  - Can be unfair, e.g., the rich man – poor man problem
  - Require convexity of the utility region
  - Independence of irrelevant alternatives axiom
  - Provide only a static solution for the problem, i.e., no discussion of the bargaining process
- Alternatives?
  - The Kalai – Smorodinsky solution
  - Dynamic bargaining and the Rubinstein process

Dynamic Bargaining

- The NBS is the static solution in the sense that we only care about the outcome
  - How about the bargaining process?

- Dynamic bargaining
  - Interested in the players interactions to reach an agreement
  - Broader than static bargaining, although linked to it
  - In this trial lecture, we cover the Rubinstein process although many others exists
- Famous one: Rubinstein Bargaining Process
  - There are other bargaining processes (a hot research topic in game theory)

Summary