Lecture 11-(a): Population Games: Motivation

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Contents

- Two examples for motivating population games
- 1. Drawing an interesting situation from a symmetric normal-form game
- 2. Non-atomic congestion game
- Generalization of population will follow soon

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Example 1: Hawk-Dove Game

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}(v-c), \frac{1}{2}(v-c)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$

- There is a resource of value v to be shared. If a player plays "Hawk," it is aggressive and will try to take the whole resource for itself. If the other player is playing "Dove," it will succeed in doing so. If both players are playing "Hawk," then they fight and they share the resource but lose c in the process. If they are both playing "Dove," then they just share the resource.

- Interpret the payoffs as corresponding to **fitness**, e.g., greater consumption of resources leads to more offspring

Hawk-Dove Game: Wait! I saw this

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}(v-c), \frac{1}{2}(v-c)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$

- Entire population is divided into the following portions depending o n their strategy choices $x = (x_h, x_d)$
  - This can be understood as a mixed strategy profile in the normal form game
- Mixed Strategy NE?
- Ah-ha! In this case, a strategy is nothing but a vector of portions of human s $x = (x_h, x_d)$, where each portion corresponds to each strategy
- A person's choice of a strategy **s** in the normal-form game == A portion of people who chooses the strategy **s**

Different Angle: Hawk-Dove Game

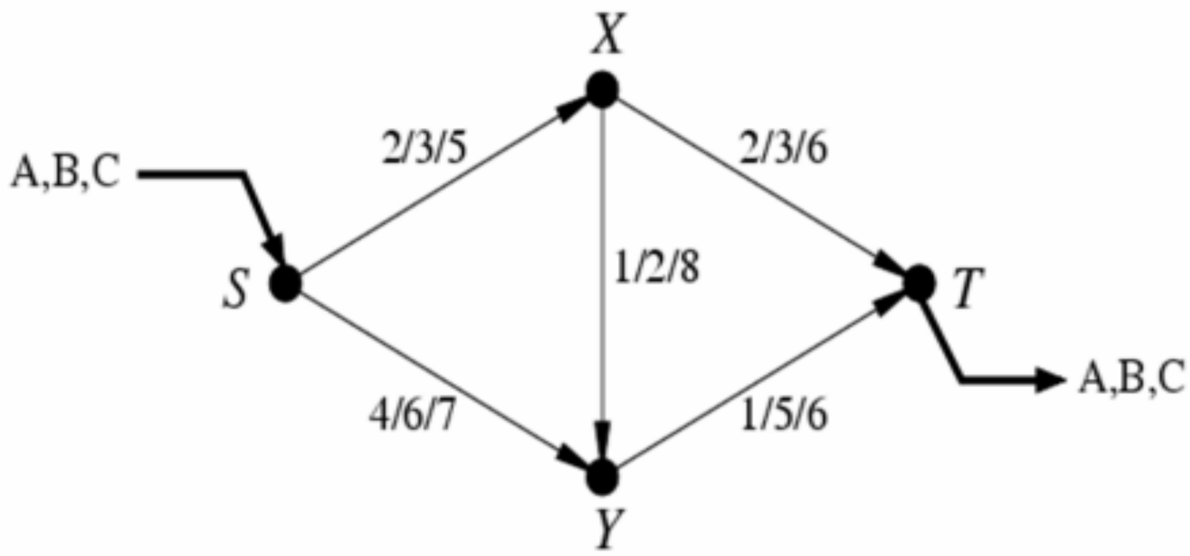
Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}(v-c), \frac{1}{2}(v-c)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$

- Consider a society consisting of "many" (mathematically infinite) humans who want to have more offspring
- Assume
  - Each infinitesimal human is matched to play the game, where each pair of humans meet exactly once.
  - Entire population is divided into the following portions depending on their strategy choices $x = (x_h, x_d)$
- What is the (average) payoff of choosing strategy "Hawk"?
  -
- What is the (average) payoff of choosing strategy "Dove"?

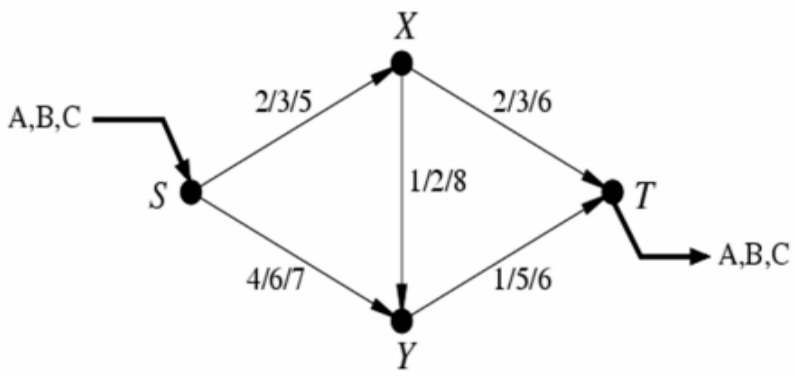
- Maybe, using this way, we can understand how animals behave in the jungle?

Another example: Congestion Game

- Three players from S to T (A,B,C)
- a/b/c: cost when one/two/three players use that road
- Total cost of each player is the aggregate link cost over its path



Earlier Model



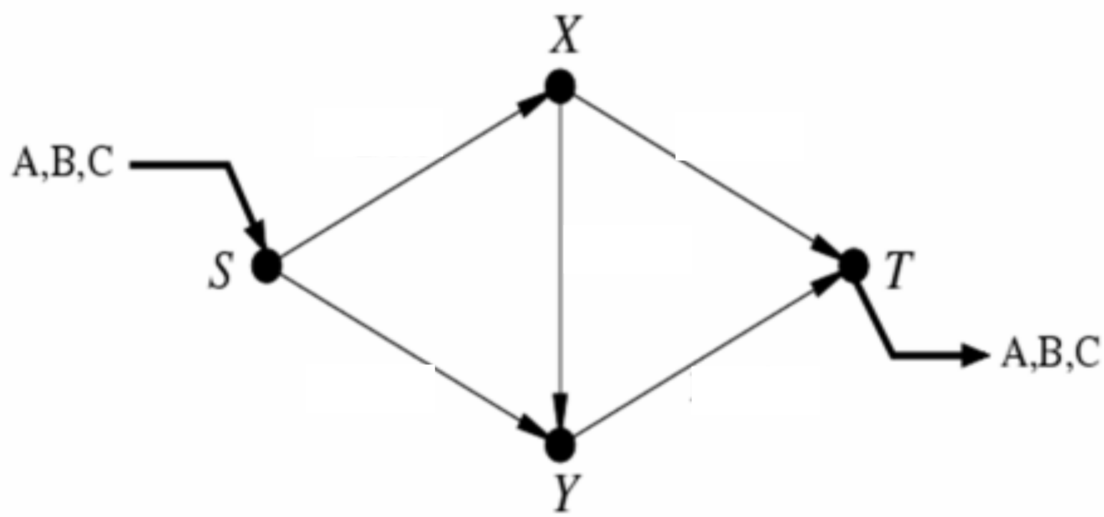
- E, a finite set of congestible elements.
- Players $i \in \{1, \ldots, n\}$, each with a strategy set S_i, where each strategy $P \in S_i$ is a subset of E. (Each strategy choice "congests" some of the congestible elements.)
- Delay functions $d_e \geq 0$ for each $e \in E$.

Further, given a set of strategy choices $P_i \in S_i$ for each player i, we defined the following:

- The *congestion* on an element e, given by $x_e = |\{i : e \in P_i\}|$, the number of players congesting that element.
- The *delay* on each element e, given by $d_e(x_e)$.
- The *cost* for each player i, equal to $\sum_{e \in P_i} d_e(x_e)$, the sum of delays for all elements used by that player.

A different, yet similar congestion game

- Three classes (or types) from S to T (A,B,C)
  - Each class is a player
  - Multiple classes are playing the game
- Each class has its mass, say r_i (equivalently the rate of traffic that needs to be delivered by class i.
- Each class selects a strategy from the strategy set S_i
- Each class of players is allowed to distribute fractionally over the strategy set



Difference between Two Games

- Unsplittable (atomic) congestion game
  - A finite number of users
  - Each user chooses a path on which he transports all of his load

- Splittable (non-atomic) congestion game
  - Treat traffic similarly to flow in a network where we can split up the load to several paths
  - Analogously, we can look it as (infinitely many users, where each user controls infinitesimal portion of the total traffic
  - This motivates a notion of "population game"

More general stories will follow

Lecture 11: Population Game and Evolutionary Game

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Contents

- Population Game: An approach to modeling recurring strategic interactions in large populations of small anonymous agents.
- Originally developed in biology to understand how the animals interact, where they have different genetic programs leading to varying levels of reproductive success.
- Key words: population game, evolutionary dynamics, ESS (Evolutionarily Stable Strategy)

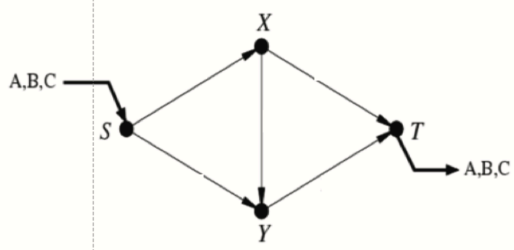
Population Game: When?

- The number of agents is large
- Individual agents are small: any single agent's behavior has no effect on others.
- Agents interact anonymously: Each agent's payoffs only depend on opponents' behavior through the distribution of their choices.
- Payoffs are continuous: Ensures that very small changes in aggregate behavior do not lead to large changes in strategies' payoffs.

Motivating Example: Non-atomic Congestion Game

Recall this game!

- Three classes (or types) from S to T (A,B,C)
  - Each class is a player
  - Multiple classes are playing the game
- Each class has its mass, say r_i (equivalently the rate of traffic that needs to be delivered by class i.
- Each class selects a strategy from the strategy set S_i
- Each class of players is allowed to distribute fractionally over the strategy set



Population Game: Population, Strategies, and States

- $\mathcal{P} = \{1, 2, 3, \ldots, \mathrm{p}\}$: A society that has p populations (or classes, or types)

  Ex) society: USA, population 1: Texas, population 2: California

  Ex) Non-atomic congestion game?

- Agents in population $p \in \mathcal{P}$ form a continuum of mass $m^p > 0$.

  Ex) Non-atomic congestion game?

- $S^p = \{1, \ldots, n^p\}$: the set of (pure) strategies available to agents in population p.

- Ex. $S^1 = \{sell, buy\}$, $S^2 = \{study, play\}$.

  Ex) Non-atomic congestion game?

- $n = \sum_{p \in \mathcal{P}} n^p$: total number of pure strategies in all populations

  Ex) Non-atomic congestion game?

- The set of population states (or strategy distributions) for population p is $X^p = \{x^p \in \mathcal{R}_+^{n^p} : \sum_{i \in S^p} x_i^p = m^p\}$.

  In population p, how many people are using each strategy?

  Ex) Non-atomic congestion game?

- The set of social states: $X = \prod_{p \in \mathcal{P}} X^p$

  Ex) Non-atomic congestion game?

- The special case when $\mathrm{p} = 1$, i.e., a single population: We omit the superscript p, and use

  X

  $S = \{1, \ldots, n\}$

Payoffs

- $F_i^p : X \mapsto \mathcal{R}$: payoff function for strategy $i \in S^p$

  Note: Payoff is not the one for each player in the earlier cases. Payoff is the one for each strategy.

- $F^p : X \mapsto \mathcal{R}^{n^p}$: payoff functions for all strategies in S^p.

- A payoff function: $F : X \mapsto \mathcal{R}^n$ is a continuous map that assigns each social state a vector of payoffs, one for each strategy in each population.

  Ex) Non-atomic congestion game?

- Note: For a single population, we just use F_i and F.

Example 1: Matching in Normal Form Games (Symmetric)

- Method 1: Obtaining a population game from this normal form game

- A symmetric two-player normal form game, represented by a matrix $A \in \mathcal{R}^{n\times n}$

	1	2	3
1	A_{11}, A_{11}	A_{12}, A_{21}	A_{13}, A_{31}
2	A_{21}, A_{11}	A_{22}, A_{22}	A_{23}, A_{32}
3	A_{31}, A_{11}	A_{32}, A_{23}	A_{33}, A_{33}

- Agents in a single population are matched to play A, with each pair of agents meeting exactly once.

- Aggregating payoffs over all matches, the payoff to strategy i for the population state x

  $F_i(x) = \sum_{j\in S} A_{ij}x_j = (Ax)_i$

- The population game is described by $F(x) = A \cdot x$

- $F(x) = A \cdot x$'s another interpretation: Each agent is randomly matched against a single opponent. Then, the expected payoff is ...

Example 2: Matching in Normal Form Games (Asymmetric)

- An asymmetric two-player normal form game, represented by a matrix $A \in \mathcal{R}^{n \times n}$, two strategy sets $S^1 = \{1, \ldots, n^1\}$, and $S^2 = \{1, \ldots, n^2\}$.

	1	2	3
1	U^1_{11}, U^2_{11}	U^1_{12}, U^2_{12}	U^1_{13}, U^2_{13}
2	U^1_{21}, U^2_{21}	U^1_{22}, U^2_{22}	U^1_{23}, U^2_{23}

- Two unit mass, each corresponding to each player role.

- Each agent is matched with every member of other population to play the game (U^1, U^2), or one can assume that each agent is randomly matched with a single member of the other population.

- The payoff functions for populations 1 and 2 are given by: $F^1(x) = U^1 x^2$ and $F^2(x) = (U^2)' x^1$.

Example 2: (Non-atomic) Congestion Game

- Method 2: Obtaining a population game from "playing in the field"

- A collection of nodes is connected by a network of links. For each ordered pair of towns there is a population of agents, each of whom needs to transfer data from the first node in the pair to the destination.

- An agent must choose a path connecting two nodes in the pair.

- A finite collection of links Φ.

- Each strategy $i \in S^p$ requires the use of some collection of links (i.e., path) $\Phi_i^p \subset \Phi$. The set $\rho^p(\phi) = \{i \in S^p : \phi \in \Phi_i^p\}$: contains those strategies in S^p that require link ϕ.

- Each link ϕ has a cost function $c_\phi : \mathcal{R}_+ \mapsto \mathcal{R}$ whose argument is the link's utilization level u_ϕ, the total mass of agents using the link, i.e.,

$$u_\phi(x) = \sum_{p \in \mathcal{P}} \sum_{i \in \rho^p(\phi)} x_i^p.$$

- Payoffs are:

$$F_i^p(x) = -\sum_{\phi \in \Phi_i^p} c_\phi(u_\phi(x))$$

- Compare the congestion game modeled by a population game approach and that modeled by a normal game (see the previous potential game lecture on congestion game).

Lecture 11-(c): Population Games: Equilibrium

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Nash Equilibrium of Population Game

- Essentially,
- It's equal to finding the mixed NEP of the game
  - Social states refer to the probabilistic distribution over the strategies o f all populations
- But, there exists other notions of equilibrium that is worth co nsidering
  - Evolutionary stable strategy

Best Response and Nash Eq.

- $b^p : X \mapsto S^p$: population p's pure best response correspondence, specifying the strategies (i.e., set-valued function) that are optimal at each social state x:

$$b^p(x) \in \arg\max_{i \in S^p} F_i^p(x).$$

Note: This is $\arg\max$ over S^p.

Note: This does NOT mean that all agents in each population chooses a singe strategy.

- Let Δ^p be the simplex in $\mathcal{R}^{n^p}$, i.e.,

$$\Delta^p = \{y^p \in \mathcal{R}_+^{n^p} : \sum_{i \in S^p} y_i^p = 1\}.$$

The mixed best response correspondence for population p, $B^p : X \mapsto \Delta^p$ is given by:

$$B^p = \{y^p \in \Delta^p : y_i^p > 0 \rightarrow i \in b^p(x)\},$$

$B^p(x)$ is the set of prob. distributions in Δ^p whose supports only contain pure strategies that are optimal at x (otherwise, contradiction!). $B^p(x)$ is the convex hull of the vertices of Δ^p corresponding to elements of $b^p(x)$.

- Social state $x \in X$ is a Nash equilibrium of the game F if each agent in every population chooses a best response to x:

$$NE(F) = \{x \in X : x^p \in m^p B^p(x) \text{ for all } p \in \mathcal{P}\}$$

Note: Here, a population state is the NE, not a strategy vector.

Note: Thus, at NE, for the given population state, the fractions of choosing "optimal" strategies are exactly equal to the fractions expressed by NE.

- Exercise 1. Find the NE of the population game defined by the following normal form game:

(1)

$$\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 2,2 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2,2 & 0,3 \\ 3,0 & 1,1 \end{pmatrix}$$

Again: Hawk-Dove Game (was a HW prob.)

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}(v-c), \frac{1}{2}(v-c)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$

- If $v > c$, then there is a unique **strict** Nash equilibrium, which is (Hawk, Hawk).
- If $v = c$, then there exists a unique Nash equilibrium, (Hawk, Hawk), though this is not a strict Nash equilibrium.
- If $v < c$, then there exists three Nash equilibria: (Hawk, Dove) and (Dove, Hawk), which are non-symmetric strict equilibria, and a mixed strategy symmetric equilibrium.

Evolution in the H-D Game

- If $v > c$, then we expect all agents to choose "Hawk". Those who do not will have lower fitness.
- A different way of thinking about the problem: imagine a population of agents playing "Dove" in this case.
- Suppose there is a **mutation**, so that one agent (or a small group of agents) starts playing "Hawk".
- This latter agent and its offspring will **invade** the population, because they will have greater fitness.
- The notion of evolutionarily stable strategies or evolutionary stability follows from this reasoning.

Invasion

- In a population game F, state y can weakly invade state x if $(y - x)'F(x) \geq 0$ (i.e., $y'F(x) \geq x'F(x)$). "Strongly" is the case of strict inequality.

- Interpretation: Consider a single population of agents who play the game F and whose initial behavior is described by state $x \in X$.

  Now imagine that a very small group of agents decide to switch strategies. After these agents select their new strategies, the distribution of choice within their group is described by $y \in X$.

  But, since the group is so small, the impact of its behavior on the overall population state is negligible (That's why we have $F(x)$ at LHS and RHS)

  Thus, the average payoff in the invading group (i.e., $y'F(x)$) is at least as high as that in the incumbent population (i.e., $x'F(x)$).

ESS (Evolutionarily Stable Strategy)

- A different solution concept from NE (So, your question: relation between ESS and NE? We will look at this later)

- In the single population game F, $x \in X$ is an ESS of F if there is a neighborhood O of x such that $y'F(y) < x'F(y)$ for all $y \in O - \{x\}$. → Locally stable

- Equivalent definition (Can you visualize why this is true?).

  There is an $\bar{\epsilon} > 0$ such that the following equation holds for all $y \in X - \{x\}$ and $\epsilon \in (0, \bar{\epsilon})$:

$$y'F(\epsilon y + (1-\epsilon)x) < x'F(\epsilon y + (1-\epsilon)x)$$

- You can find the following definitions in other books,

  Consider a two player, symmetric strategic form game with the payoff function u. A (possibly mixed) strategy is $\sigma \in \Sigma$, where Σ is the set of all mixed strategies.

  Definition. A strategy $\sigma^\star \in \Sigma$ is evolutionarily stable if there exists $\bar{\epsilon} > 0$, such that for any $\sigma \neq \sigma^\star$ and for any $\epsilon < \bar{\epsilon}$, we have

$$u(\sigma^\star, \epsilon\sigma + (1-\epsilon)\sigma^\star) > u(\sigma, \epsilon\sigma + (1-\epsilon)\sigma^\star)$$

- Consider a population game from a symmetric two-player normal-form game G. Then, from the properties of expectation,

$$(1-\epsilon)u(\sigma^\star,\sigma^\star)+\epsilon u(\sigma^\star,\sigma) > (1-\epsilon)u(\sigma,\sigma^\star)+\epsilon u(\sigma,\sigma)$$

- The above only needs to hold for small ϵ, this is equivalent requiring that either

$$u(\sigma^\star,\sigma^\star) > u(\sigma,\sigma^\star), \tag{1}$$

or else,

$$\begin{aligned} u(\sigma^\star,\sigma^\star) &= u(\sigma,\sigma^\star), \\ u(\sigma,\sigma^\star) &> u(\sigma,\sigma). \end{aligned} \tag{2}$$

- Thus, ESS implies NE.

- Interpretation: An evolutionarily stable strategy σ^* is a Nash equilibrium. If σ^* is not a strict Nash equilibrium, then any other strategy σ that is a best response to σ^* must be worse against itself than against σ^*.

Homework: Prove that the above things are equivalent.

ESS and NE

Theorem

- *A strict (symmetric) Nash equilibrium of a symmetric game is an evolutionarily stable strategy.*
- *An evolutionarily stable strategy is a Nash equilibrium.*

Monomorphic vs. Polymorphic

- Monomorphic. Populations whose members all choose the same strategy—but allowed this common strategy to be a mixed strategy.

  Comparison between two different (mixed) strategies.

- Polymorphic. Mass (or number of people) which chooses a pure strategy i for each strategy i.

  Comparison between two different populations (incumbent and invading).

Back to: Hawk-Dove Game

Player 1/Player 2	Hawk	Dove
Hawk	$\left(\frac{1}{2}(v-c), \frac{1}{2}(v-c)\right)$	$(v, 0)$
Dove	$(0, v)$	$\left(\frac{1}{2}v, \frac{1}{2}v\right)$