

Lanada

# Lecture 9: Repeated Game

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# Basic Terminologies

- **Dynamic/Static Game**
  - Game in which we have sequence of moves or not
- **Complete/Incomplete Information**
  - Games in which the strategy space and player's payoff functions are common knowledge, or not
- **Perfect/Imperfect Information**
  - Each move in the game the player with the move knows the full *history* of the game thus far.
  - At some move the player with the move does not know the history of the game.
  - Typically used for dynamic games

# Prisoner's Dilemma

	Cooperate	Defect
Cooperate	1, 1	-1, 2
Defect	2, -1	0, 0

- What about playing this game iteratively?
- Iterative Prisoner's Dilemma
- A special form of dynamic games
- Difference from the earlier extensive form game, or stackelberg game?

# Motivation

	Cooperate	Defect
Cooperate	1, 1	-1, 2
Defect	2, -1	0, 0

- Consider this strategy S:
  - Start with C, and choose C as long as the other player chooses C
  - If in any period the other player chooses D, then choose D in *every subsequent period*
- Outcome example
  - (C,C), (C,C), (C,C) ... (C,D) (D,D) (D,D) ...
  - Why player 2 can choose D in (C,D)? For a short-term gain
  - The strategy S means that I will punish you if you defect!
- If a player value the present more highly than the future, she may or may not choose defect.
  - How *patient* is a player?

# What is Repeated Game?

	$L_2$	$R_2$
$L_1$	1,1	5,0
$R_1$	0,5	4,4

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	$L_2$	$R_2$
$L_1$	1,1	5,0
$R_1$	0,5	4,4

.....

- A game is repeated multiple times, say 10 times.
  - 한번만 할 것이라면,  $L_i$ 를 선택하겠지만, 여러 번 할 것이라면,  $R_i$ 를 선택하여서, “협조”를 구해보는 것도...
  - 만약, 협조하다가 배신하면, 내가 너를 “응징”하리...

# What is Repeated Game?

	$L_2$	$R_2$
$L_1$	1,1	5,0
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	$L_2$	$R_2$
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$R_1$	0,5	4,4

.....

- In ongoing relationships, the promise of future rewards and the threat of future punishments **may** sometimes provide **incentives for good behavior** (i.e., cooperation) today. (Nobel prize!)
- T: Period of a repeated game
  - Finite case
  - Infinite case

# Terminologies

- **Repeated games:** given a simultaneous-move game  $G$ , a repeated game of  $G$  is an extensive game with perfect information and simultaneous moves in which a history is a sequence of action profiles in  $G$ . I will denote the repeated game, if repeated  $T$  times, as  $G^T$ .
- $G$  is often called a **stage game**, and  $G^T$  is called a **supergame**.

## Payoffs of a repeated game

- A player gets a payoff from each stage game, so her total payoff from the supergame is the discounted sum of the payoffs from each stage game.
- Let's call a sequence (with  $T$  periods) of action profiles as  $(a_1, a_2, \dots, a_T)$ , then a player  $i$ 's total payoff from this sequence, when her discount factor is  $\delta$ , is

$$u_i(a_1) + \delta u_i(a_2) + \delta^2 u_i(a_3) + \dots + \delta^{T-1} u_i(a_T) = \sum_{t=1}^T \delta^{t-1} u_i(a_t).$$

- If the sequence is infinite, then the discounted sum is

$$\sum_{t=1}^{\infty} \delta^{t-1} u_i(a_t).$$



# Discounting: Rationale

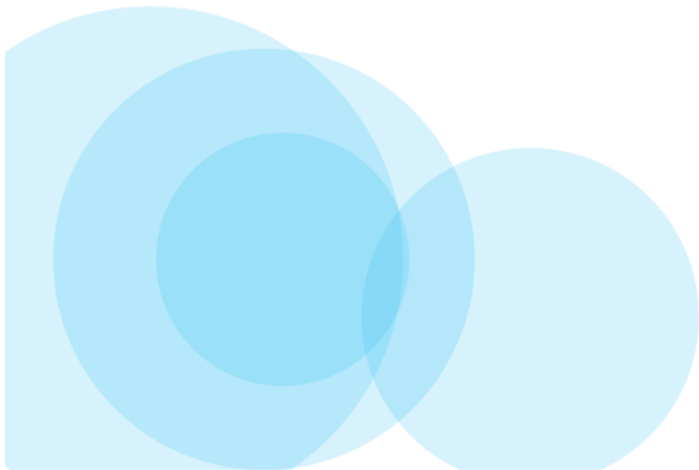
- Discounting: people value \$100 tomorrow less than \$100 today (Why? Interest rate; the world may end tomorrow.)
- The **discount factor**  $\delta$  ( $0 \leq \delta \leq 1$ ) denotes how much a future payoff is valued at the current period, or how patient a player is.
  - ▶ If a player has a  $\delta$  of .8, then \$100 tomorrow is equivalent to \$80 today for her.
- Or, I continue this game with probability  $\delta$  and end this game with probability  $1 - \delta$

# Dividing Cases

- Finite repeated game
  - (Case 1) A stage game has unique NE
  - (Case 2) A stage game has multiple NEs
- (Case 3) Infinite repeated game
- In this lecture, in most of examples,
  - Use Prisoner's dilemma as a stage game
- Assumptions
  - Perfect monitoring: At each period, the outcomes of all past periods are observed by all players

Finite Repeated Game:

Unique NE in the stage game



# Finitely-repeated PD

	C	D
C	2,2	0,3
D	3,0	1,1

- What happens if this game was played  $T < \infty$  times?
- We first need to decide what the equilibrium notion is. Natural choice, **subgame perfect Nash equilibrium (SPE)**.
- Recall:  $\text{SPE} \iff$  backward induction.
- Therefore, start in the last period, at time  $T$ . What will happen?

## Cont'd

- In the last period, “defect” is a dominant strategy regardless of the history of the game. So the subgame starting at  $T$  has a dominant strategy equilibrium:  $(D, D)$ .
- Then move to stage  $T - 1$ . By backward induction, we know that at  $T$ , no matter what, the play will be  $(D, D)$ . Then given this, the subgame starting at  $T - 1$  (again regardless of history) also has a dominant strategy equilibrium.
- With this argument, we have that there exists a unique SPE:  $(D, D)$  at each date.
- In fact, this is a special case of a more general result.

# (D,D,D,D): SPNE Checking

	C	D
C	2,2	0,3
D	3,0	1,1

Period:	1	...	$t-1$	$t$	$t+1$	...	$T$
$(s_1, s_2)$ :	$a^1$	...	$a^{t-1}$	$(C, X)$	$(D, D)$	...	$(D, D)$
Relation between player 1's payoffs:	$\parallel$	...	$\parallel$	$\wedge$	$\wedge$	...	$\wedge$
$(s'_1, s_2)$ :	$a^1$	...	$a^{t-1}$	$(D, X)$	$(D, ?)$	...	$(D, ?)$

## Thus, we have ...

### Theorem

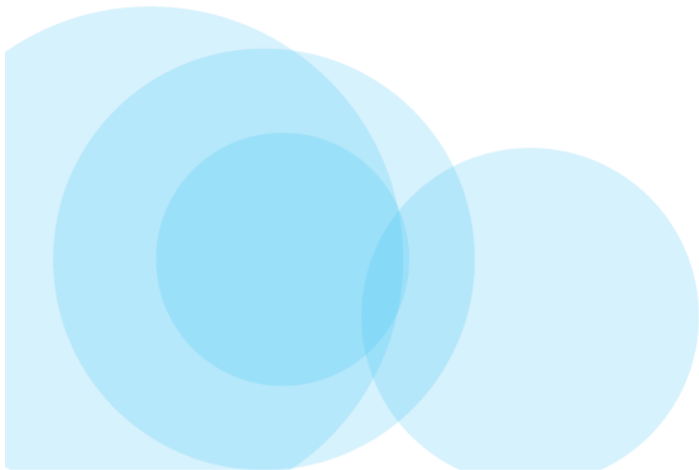
*Consider repeated game  $G^T(\delta)$  for  $T < \infty$ . Suppose that the stage game  $G$  has a unique pure strategy equilibrium  $a^*$ . Then  $G^T$  has a unique SPE. In this unique SPE,  $a^t = a^*$  for each  $t = 0, 1, \dots, T$  regardless of history.*

**Proof:** The proof has exactly the same logic as the prisoners' dilemma example. By backward induction, at date  $T$ , we will have that (regardless of history)  $a^T = a^*$ . Given this, then we have  $a^{T-1} = a^*$ , and continuing inductively,  $a^t = a^*$  for each  $t = 0, 1, \dots, T$  regardless of history.

- In this case, no cooperation appear even if we repeat the game.

## Infinite Repeated Game:

What happens if we don't know when the game would end?





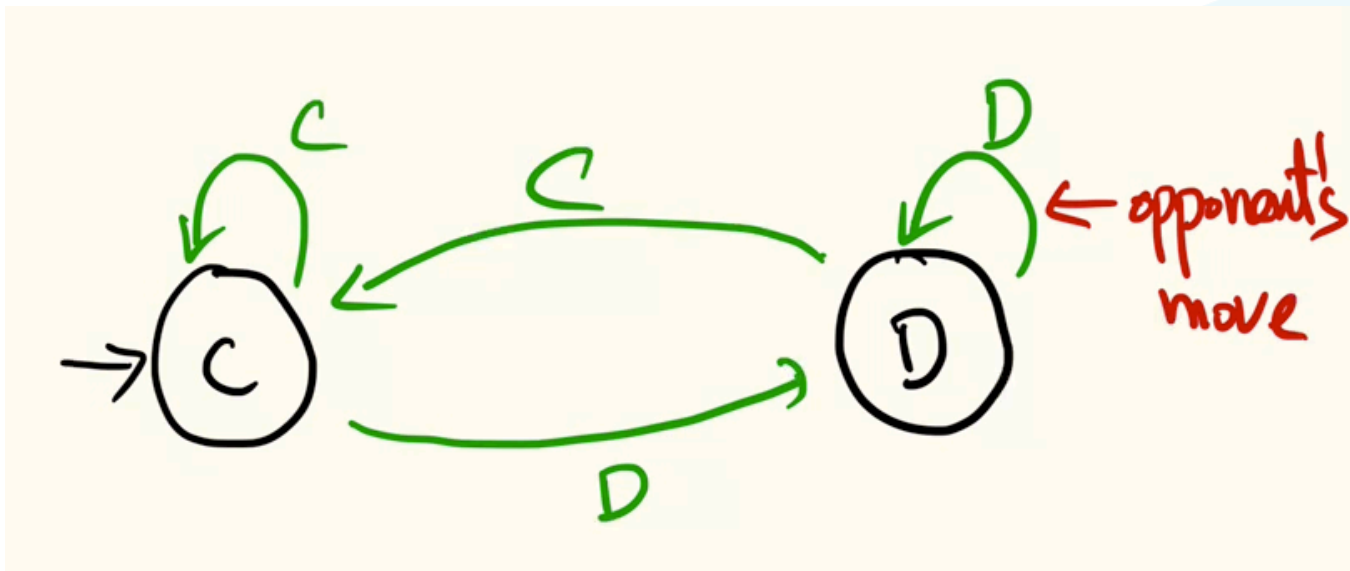
# Intro

- An infinitely repeated game usually has more SPNEs than a finitely repeated game, and it may have multiple SPNEs even if the stage game has a unique NE.
- But playing the NE strategies in each stage game, regardless of history, is still a SPNE in the infinitely repeated game.
  - ▷ So each player choosing defection is a SPNE in infinitely repeated PD as in finitely repeated PD.
  - ▷ But there are other SPNEs in infinitely repeated PD.
- Our interest: Strategies inducing cooperation is SPNE or not

# Strategy: Tit-for-tat

	C	D
C	2,2	0,3
D	3,0	1,1

- TFT strategy
  - Play C first,
  - Then, do whatever the other play did in the previous period



# What does TFT do when playing against these strategies?

	C	D
C	2,2	0,3
D	3,0	1,1

Strategies

	always defect	always cooperate	C-D-D-D-...	C-D-C-D-...
always defect			●	
always cooperate		●		
TFT		●		
D-C-D-C-D-...				●

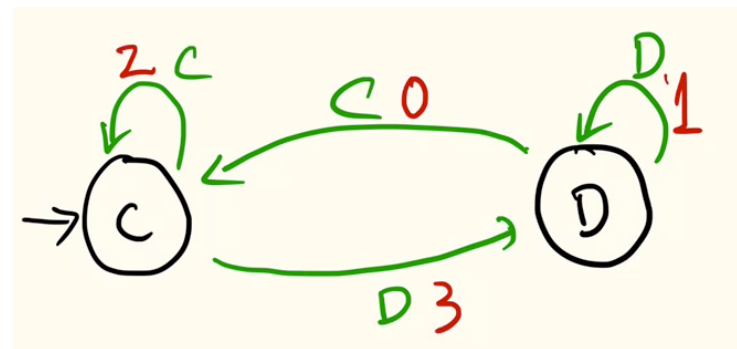
# Which is better response to TFT?

	C	D
C	2,2	0,3
D	3,0	1,1

- Always D = D, D, D, D, ...
  - Payoff =  $3 + \delta/(1 - \delta)$
  - Good for low  $\gamma$
- Always C = C, C, C, C, ...
  - Payoff =  $2 * 1/(1 - \delta)$
  - Good for high  $\gamma$
- For what value of  $\gamma$ , are they equally good?  $1/2$

# Best Response to Some Strategies ( $\gamma > \frac{1}{2}$ )

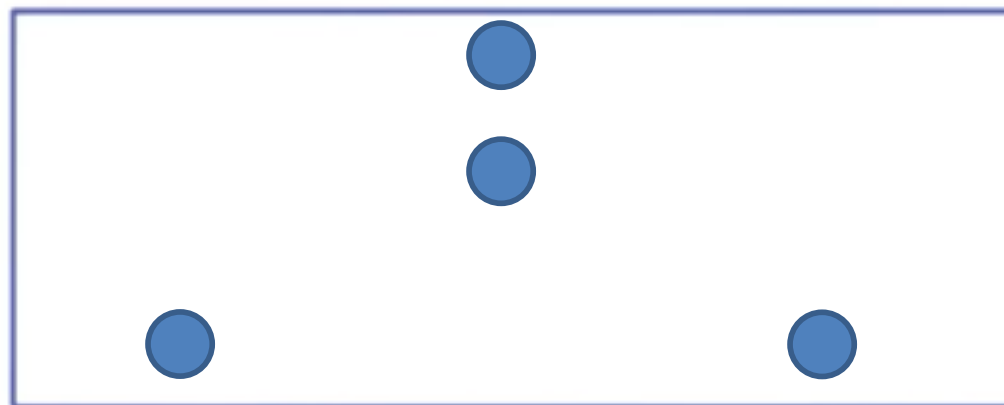
- State: opponent choice
- Edge: my choice
- Edge label: my payoff



Given strategies

always C  
 always D  
 TFT

always C      always D      TFT

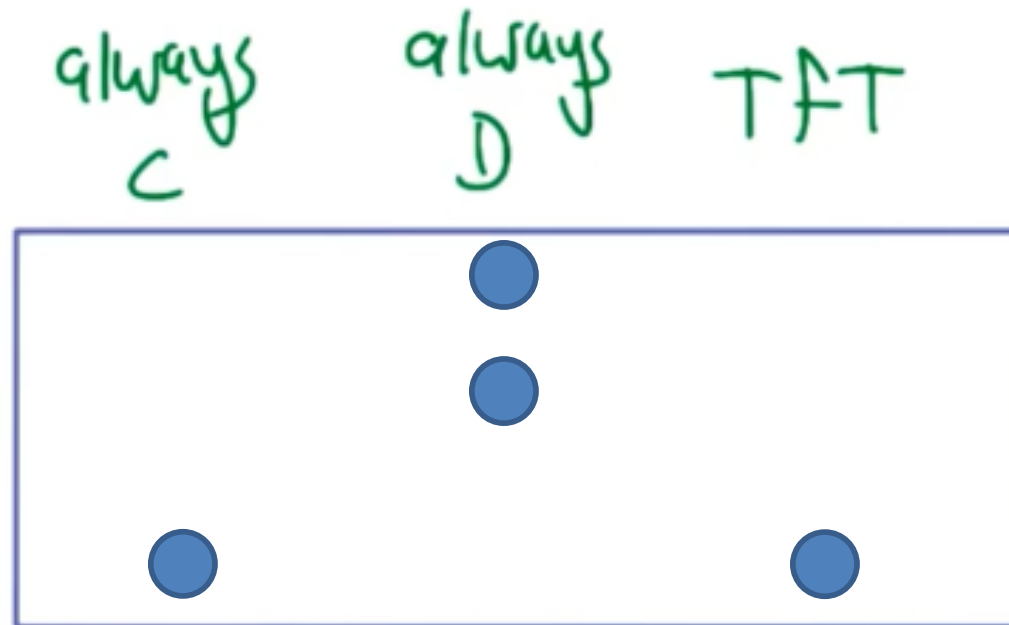


What are NEPs? Is there cooperation in some NEP?

# Cooperation in Infinite Repeated Game

Given strategies

always C  
 always D  
 TFT



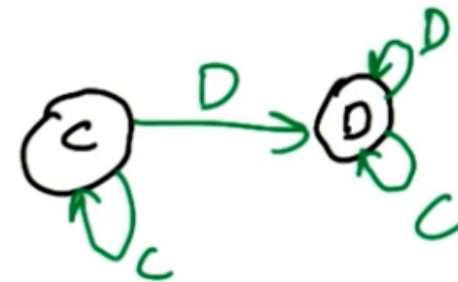
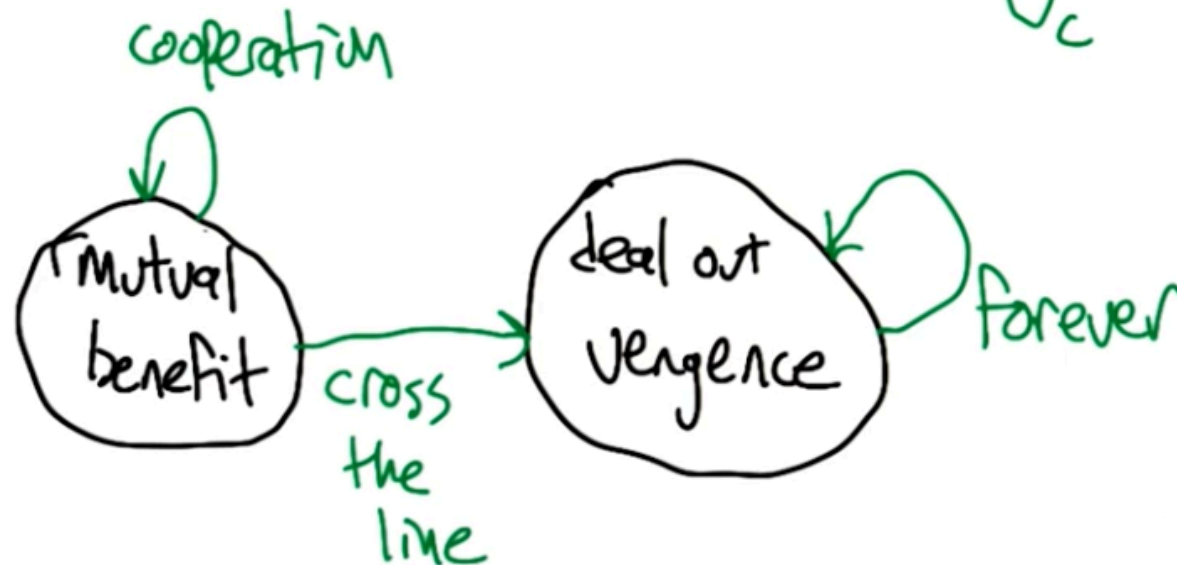
What are NEPs? Is there cooperation in some NEP? Why?

TFT is still a NEP for small  $\delta$ ?

# Strategy: Grim Trigger (GT)

- Start with C
- Then, play C if opponent has played D, and play D otherwise
  - Draconian policy

Grim Trigger



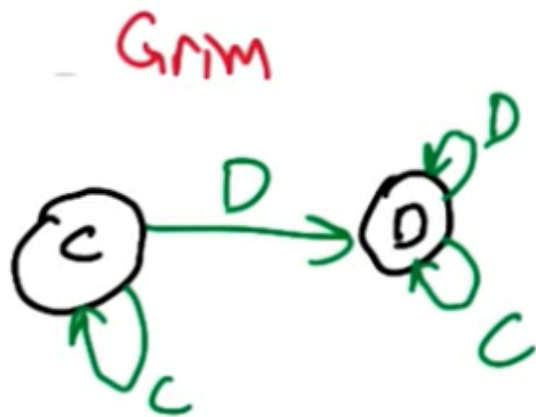
# Grim Trigger: NEP

- Under what conditions of  $\delta$ , (GT,GT) is a NEP?

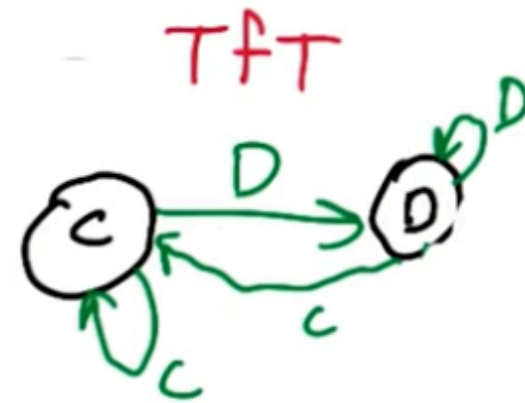


# Incredible Threat (NE but not SPNE)

- Consider a strategy (GT, TFT) as follows:



VS.



- Is this NE?
- Is this SPNE?

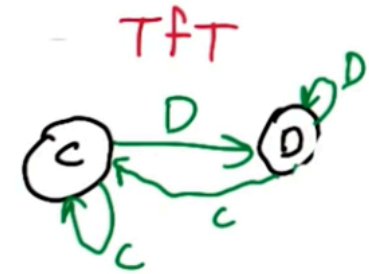
## Checking SPNE Easily: One Deviation Property

- **One-deviation property:** no player can increase her payoff by changing her action at the start of any subgame in which she is the first-mover, given the other players' strategies *and* the rest of her own strategy.
- A strategy profile in an infinitely repeated game with a discount factor less than 1 is a SPNE if and only if it satisfies the one-deviation property.

# TFT: SPNE

- Under what  $\delta$ , (TFT,TFT) is a SPNE?

	C	D
C	2,2	0,3
D	3,0	1,1

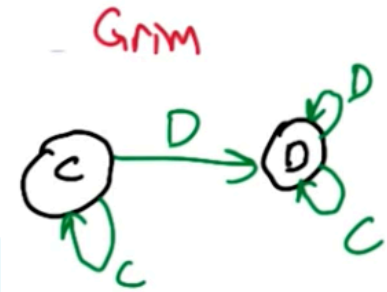




# Grim Trigger Strategy: SPNE

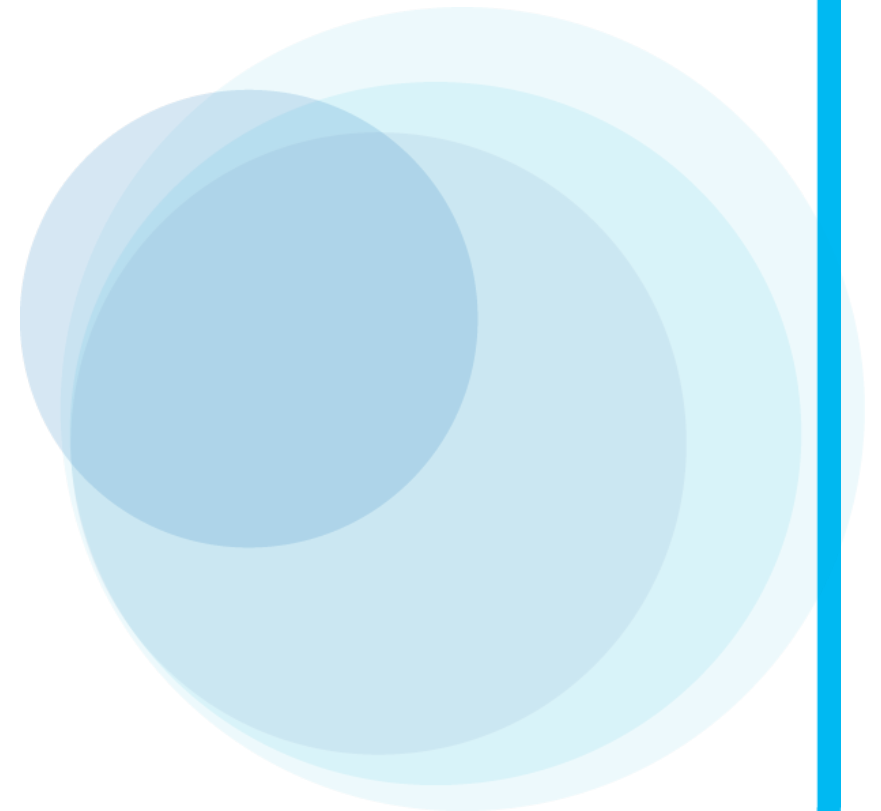
	C	D
C	2,2	0,3
D	3,0	1,1

- Under what  $\delta$ , (GT,GT) is a SPNE?



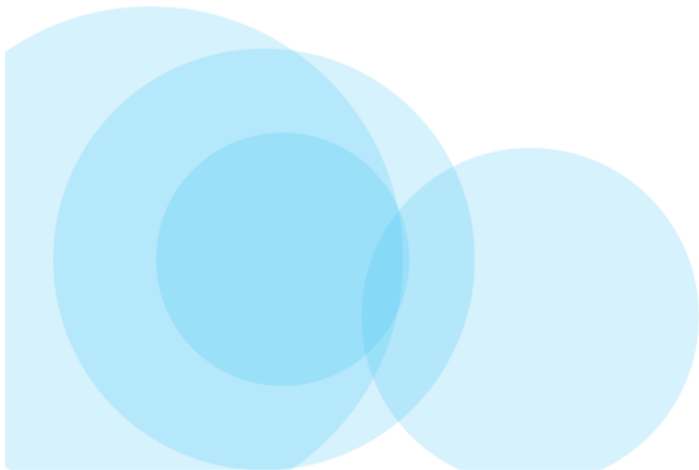
# Summary

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Finite Repeated Game:

Multiple NEs in the stage game

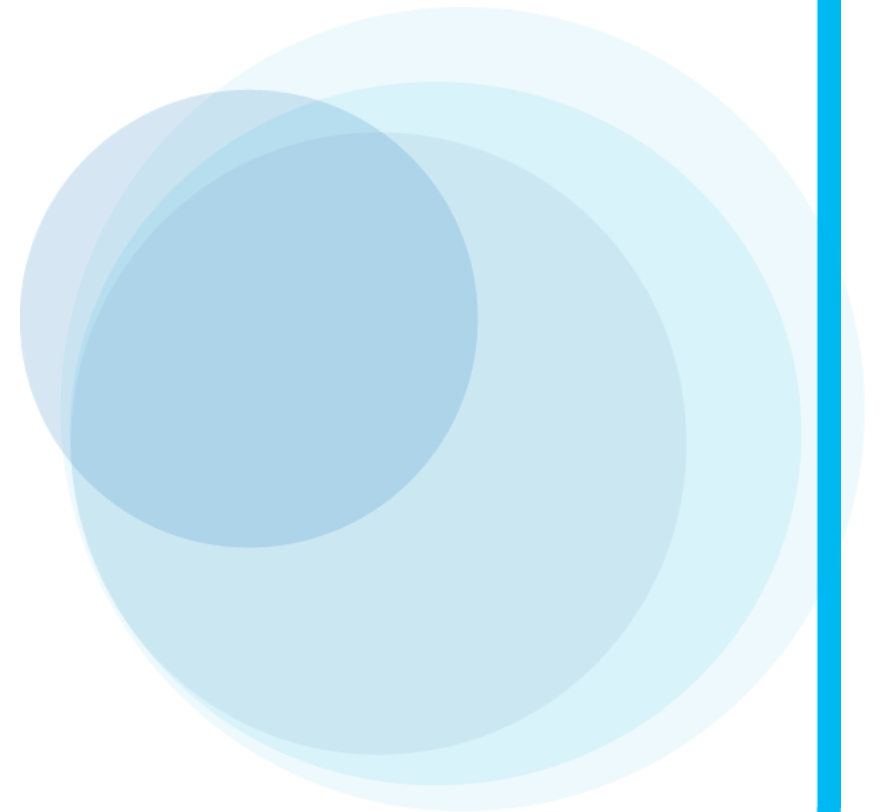




# Message

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- SPNE can include a strategy (at some stage game) that is NOT NE of the associated stage game.
- Generally, many SPNEs





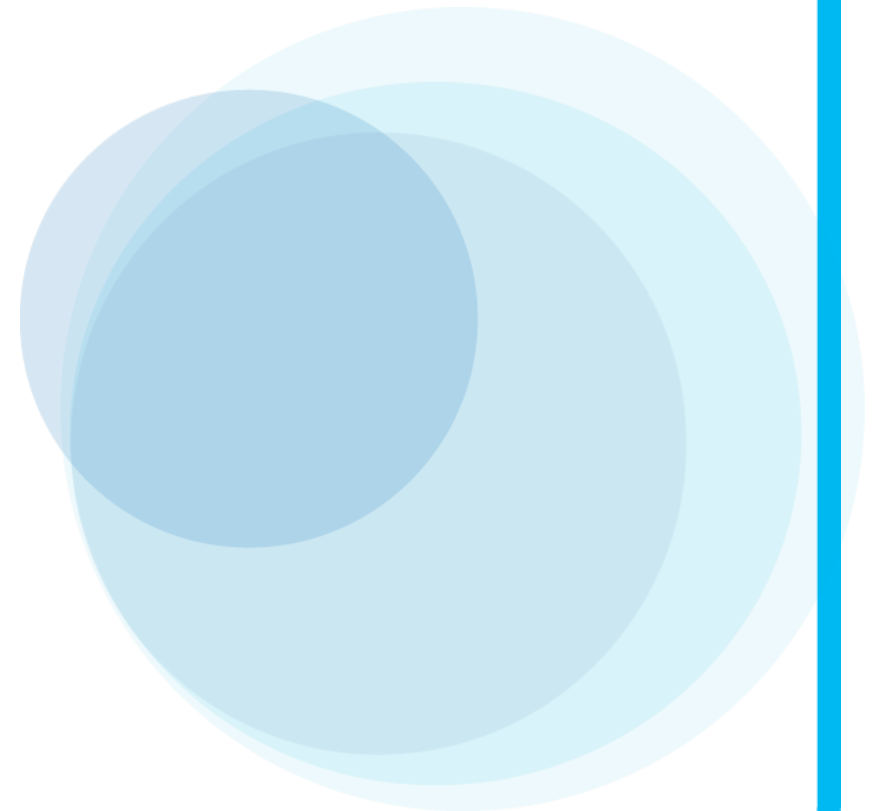
## Example: Extended PD

	Cooperate	Defect	Punish
Cooperate	4,4	0,5	0,0
Defect	5,0	1,1	0,0
Punish	0,0	0,0	3,3

- Cooperate = Quiet (묵비권),
- Defect (배반) = Fink (고자질)
- What are the NEs? (D,D) and (P,P)
- Play twice, i.e.,  $T=2$
- We will see
  - Even for the known ends, still cooperation helps
  - “Like to sustain (C,C)”, which is not an NE of the one-shot game

# Strategy “Yung” that is SPNE

- Strategy “Yung”
  - Play C and then
  - Play P if (C,C), and Play D otherwise
- Is “Yung” a strategy?
- Is “Yung” a SPNE? Yes!

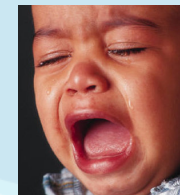


## How to check a strategy is SPNE?

- One deviation principle, i.e.,
- “Assuming that other players are playing Yung, what happens if I deviate?”
- If I get a larger payoff (by deviation), than Yung is not an SPNE .
- If I get a smaller payoff (by deviation), than Yung is an SPNE.

# What happens if I deviate?

- I deviate?
  - In other words, I don't play C, but D (no reason to play P)
  - Why? Temptation to cheat because of an increasing payoff present (현실에 눈이 어두워서...)
- If I play C (i.e., playing Yung)
  - $C \rightarrow 4 (C,C) + 3 (P,P) = 7$
- If I deviate and play D (i.e., playing some other strategy)
  - $D \rightarrow 5 (D,C) + 1 (D,D) = 6$
- Temptation to cheat ( $5-4 = 1$ ) < reward – punishment ( $3-1 = 2$ )  $\rightarrow$  I should not have deviated
- (C,C) is reward, and (B,B) is punishment
- Yung is a SPNE
  - C,P,P,P,P....



# One Deviation Property

- **One-deviation property:** no player can increase her payoff by changing her action at the start of any subgame in which she is the first-mover, given the other players' strategies *and* the rest of her own strategy.
- One-deviation property of SPNE of finite horizon games: A strategy profile in an extensive game with perfect information and a finite horizon is a SPNE if and only if it satisfies the one-deviation property.

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- A strategy profile in an infinitely repeated game with a discount factor less than 1 is a SPNE if and only if it satisfies the one-deviation property.
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