Lecture 6:
Playing with Equilibrium

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Existence and Uniqueness

- Sufficient conditions for existence and uniqueness

- This lecture
  - A little bit mathematical
  - But, we focus on just intuitions and results, rather than rigorous proofs

- Note
  - Many NE existence and uniqueness proofs do NOT directly follow thes e sufficient conditions, but they sometimes use other techniques (e.g., directly follow the definition of NE, etc).
  - But, it's useful to know that there exist these sufficient conditions.

Existence

- Mathematically,
  - Existence of NE: a fixed-point problem
  - Why?

Definition 9 *The* best response function $b_i(\mathbf{s}_{-i})$ *of a player* i *to the profile of strategies* $\mathbf{s}_{-i}$ *is a set of strategies for that player such that*

$$b_i(\mathbf{s}_{-i}) = \{s_i \in \mathcal{S}_i \mid u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s_i', \mathbf{s}_{-i}),\ \forall s_i' \in \mathcal{S}_i\}. \tag{3.5}$$

Proposition 1 *A strategy profile* $\mathbf{s}^* \in \mathcal{S}$ *is a* Nash equilibrium *of a noncooperative game if and only if every player's strategy is a* best response *to the other players' strategies, that is:*

$$s_i^* \in b_i(\mathbf{s}_{-i}^*) \text{ for every player } i. \tag{3.6}$$

Existence

- Many sufficient conditions for existence based on the existen ce of the solution of FPT (fixed point theorems).

- In the previous homework,
  - Pricing-congestion game
  - Infinitely many pure strategy space
  - Depending on the situation, pure strategy NEs may or may not exist.

Finite Game

- Pure strategy (Prisoner's Dilemma, Matching Pennies etc)
  - NE may exists
  - NE can be unique
  - No NE

P2

P1		Head	Tail
	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- Mixed Strategy

Theorem

(Nash) *Every finite game has a mixed strategy Nash equilibrium.*

- What about infinite games?

Finite Potential Game

Theorem (Rosenthal (73))

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

- Do you remember?

Pure NE Existence for Infinite Game

Theorem

(Debreu, Glicksberg, Fan) *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *such that for each* $i \in \mathcal{I}$

- S_i *is compact and convex;*
- $u_i(s_i, s_{-i})$ *is continuous in* s_{-i}*;*
- $u_i(s_i, s_{-i})$ *is continuous and concave in* s_i *[in fact quasi-concavity suffices].*

Then a pure strategy Nash equilibrium exists.

Existence of Mixed NE for Infinite Game

Theorem

(Glicksberg) *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *such that for each* $i \in \mathcal{I}$

- S_i *is a nonempty and compact metric space;*
- $u_i\,(s_i, s_{-i})$ *is continuous in* s.

Then a mixed strategy Nash equilibrium exists.

Infinite Potential Game

- Theorem (existence)
  - For infinite potential game (with a finite number of players, a pure strategy NE exists if:
    - S_i are compact
    - The potential function Φ is upper semi-continuous on S

- Theorem (uniqueness)
  - For infinite potential game (with a finite number of players, a pure strategy NE exists if:
    - S_i are compact and convex
    - The potential function Φ is is a continuously differentiable on the interior of S and concave on S

Aside: For infinite potential games

- What is the condition for a game to be a potential game?

Proposition

Let G be a game such that $S_i \subseteq \mathbb{R}$ and the payoff functions $u_i : S \to \mathbb{R}$ are continuously differentiable. Let $\Phi : S \to \mathbb{R}$ be a function. Then, Φ is a potential for G if and only if Φ is continuously differentiable and

$$\frac{\partial u_i(s)}{\partial s_i} = \frac{\partial \Phi(s)}{\partial s_i} \qquad \text{for all } i \in \mathcal{I} \text{ and all } s \in S.$$

Uniqueness of NE for General Games

- If the best response can be computed with a closed form
  - When players' utilities are concave, Rosen [1965]
- If the best responses are known with a closed form
  - Find the intersection points (generally applied in many cases)
- Just do it and try to prove from your gut feeling
  - For example, suppose that we have two NEPs.
  - Assume they are different → find a contradiction

Summary

Better-reply Secure (BRS) [Reny 1999]

- Generally, the FPT conditions are well known for the case when the utility functions are continuous w.r.t. the strategy profiles
- This condition is a generalized version of those.

Theorem

(Debreu, Glicksberg, Fan) *Consider a strategic form game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *such that for each* $i \in \mathcal{I}$

- S_i *is compact and convex;*
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BRS

Then a pure strategy Nash equilibrium exists.

Recursive Diagonally Transferable Continuous (RDTC) Games

- This is necessary & sufficient condition
- [Tian 2009] Very recently developed
- Very nice to prove that there does not exist NE
- Sufficiency: much more complex than, for example, quasi-concavity, may not be very, very useful, but we don't know

Potential and Supermodular games

- Do you remember that we have already talked some existenc e and uniqueness conditions?