

Lecture 6: Playing with Equilibrium

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Existence and Uniqueness

- Sufficient conditions for existence and uniqueness
- This lecture
 - A little bit mathematical
 - But, we focus on just intuitions and results, rather than rigorous proofs
- Note
 - Many NE existence and uniqueness proofs do NOT directly follow these sufficient conditions, but they sometimes use other techniques (e.g., directly follow the definition of NE, etc).
 - But, it's useful to know that there exist these sufficient conditions.

Existence

- Mathematically,
 - Existence of NE: a fixed-point problem
 - Why?

Definition 9 *The best response function $b_i(\mathbf{s}_{-i})$ of a player i to the profile of strategies \mathbf{s}_{-i} is a set of strategies for that player such that*

$$b_i(\mathbf{s}_{-i}) = \{s_i \in \mathcal{S}_i \mid u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s'_i, \mathbf{s}_{-i}), \forall s'_i \in \mathcal{S}_i\}. \quad (3.5)$$

Proposition 1 *A strategy profile $\mathbf{s}^* \in \mathcal{S}$ is a Nash equilibrium of a noncooperative game if and only if every player's strategy is a best response to the other players' strategies, that is:*

$$s_i^* \in b_i(\mathbf{s}_{-i}^*) \text{ for every player } i. \quad (3.6)$$

Existence

- Many sufficient conditions for existence based on the existence of the solution of FPT (fixed point theorems).
- In the previous homework,
 - Pricing-congestion game
 - Infinitely many pure strategy space
 - Depending on the situation, pure strategy NEs may or may not exist.

Finite Game

- Pure strategy (Prisoner's Dilemma, Matching Pennies etc)
 - NE may exist
 - NE can be unique
 - No NE

		P2	
		Head	Tail
P1	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- Mixed Strategy

Theorem

(Nash) *Every finite game has a mixed strategy Nash equilibrium.*

- What about infinite games?

Finite Potential Game

Theorem (Rosenthal (73))

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

- Do you remember?

Pure NE Existence for Infinite Game

Theorem

(Debreu, Glicksberg, Fan) Consider a strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ such that for each $i \in \mathcal{I}$

- S_i is compact and convex;
- $u_i(s_i, s_{-i})$ is continuous in s_{-i} ;
- $u_i(s_i, s_{-i})$ is continuous and concave in s_i [in fact quasi-concavity suffices].

Then a pure strategy Nash equilibrium exists.

Existence of Mixed NE for Infinite Game

Theorem

(Glicksberg) Consider a strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ such that for each $i \in \mathcal{I}$

- S_i is a nonempty and compact metric space;
- $u_i(s_i, s_{-i})$ is continuous in s .

Then a mixed strategy Nash equilibrium exists.

Infinite Potential Game

● Theorem (existence)

- For infinite potential game (with a finite number of players, a pure strategy NE exists if:
 - S_i are compact
 - The potential function Φ is upper semi-continuous on S

● Theorem (uniqueness)

- For infinite potential game (with a finite number of players, a pure strategy NE exists if:
 - S_i are compact and convex
 - The potential function Φ is continuously differentiable on the interior of S and concave on S

Aside: For infinite potential games

- What is the condition for a game to be a potential game?

Proposition

Let G be a game such that $S_i \subseteq \mathbb{R}$ and the payoff functions $u_i : S \rightarrow \mathbb{R}$ are continuously differentiable. Let $\Phi : S \rightarrow \mathbb{R}$ be a function. Then, Φ is a potential for G if and only if Φ is continuously differentiable and

$$\frac{\partial u_i(s)}{\partial s_i} = \frac{\partial \Phi(s)}{\partial s_i} \quad \text{for all } i \in \mathcal{I} \text{ and all } s \in S.$$

Uniqueness of NE for General Games

- If the best response can be computed with a closed form
 - When players' utilities are concave, Rosen [1965]
- If the best responses are known with a closed form
 - Find the intersection points (generally applied in many cases)
- Just do it and try to prove from your gut feeling
 - For example, suppose that we have two NEPs.
 - Assume they are different → find a contradiction

Summary

Better-reply Secure (BRS) [Reny 1999]

- Generally, the FPT conditions are well known for the case when the utility functions are continuous w.r.t. the strategy profile s
- This condition is a generalized version of those.

Theorem

(Debreu, Glicksberg, Fan) Consider a strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ such that for each $i \in \mathcal{I}$

- S_i is compact and convex;
- $u_i(s_i, s_{-i})$ is continuous in s_{-i} ;
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BRS

Then a pure strategy Nash equilibrium exists.

Recursive Diagonally Transferable Continuous (RDTC) Games

- This is **necessary & sufficient** condition
- [Tian 2009] Very recently developed
- Very nice to prove that there does not exist NE
- Sufficiency: much more complex than, for example, quasi-concavity, may not be very, very useful, but we don't know

Potential and Supermodular games

- Do you remember that we have already talked some existence and uniqueness conditions?

