Lecture 5:
Potential Game

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Contents

- There exists some special games with "beautiful" properties
- One great example is "Potential Game"
- What "beautiful" properties?
- What is Potential Game?

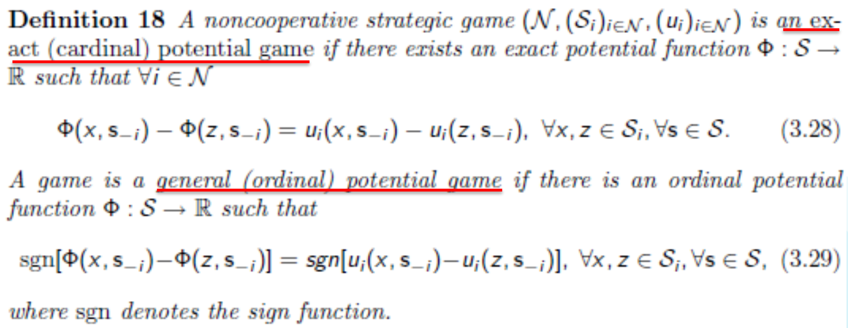
Potential games

Potential Game: Summary

- A special class of non-cooperative games having a special structure
- The variations of the users' utilities can be captured by a single function known as the **potential function**
- Potential games are characterized by their simplicity and the existence or uniqueness of a Nash equilibrium solution
- Often, potential games are useful when dealing with continuous-kernel games

Potential Game

- Formally,



Definition 18 *A noncooperative strategic game $(\mathcal{N},(\mathcal{S}_i)_{i\in\mathcal{N}},(u_i)_{i\in\mathcal{N}})$ is an exact (cardinal) potential game if there exists an exact potential function $\Phi:\mathcal{S}\rightarrow\mathbb{R}$ such that $\forall i\in\mathcal{N}$*

$$\Phi(x,s_{-i})-\Phi(z,s_{-i})=u_i(x,s_{-i})-u_i(z,s_{-i}),\ \forall x,z\in\mathcal{S}_i,\forall s\in\mathcal{S}. \qquad (3.28)$$

A game is a general (ordinal) potential game if there is an ordinal potential function $\Phi:\mathcal{S}\rightarrow\mathbb{R}$ such that

$$\mathrm{sgn}[\Phi(x,s_{-i})-\Phi(z,s_{-i})]=\mathrm{sgn}[u_i(x,s_{-i})-u_i(z,s_{-i})],\ \forall x,z\in\mathcal{S}_i,\forall s\in\mathcal{S}, \qquad (3.29)$$

where sgn denotes the sign function.

- In *exact potential games, the difference in individual utilities* achieved by each player when changing *unilaterally its strategy has* the same value as the difference in values of the potential function. In *ordinal potential games, only* the signs *of the differences have to be the same.*

Example: Prisoner's Dilemma

- A potential function assigns a real value for every $s \in S$.
- Thus, when we represent the game payoffs with a matrix (in finite games), we can also represent the potential function as a matrix, each entry corresponding to the vector of strategies from the payoff matrix.

Example

The matrix P is a potential for the "Prisoner's dilemma" game described below:

$$G=\begin{pmatrix}(1,1) & (9,0)\\ (0,9) & (6,6)\end{pmatrix}, \qquad P=\begin{pmatrix}4 & 3\\ 3 & 0\end{pmatrix}$$

Pure Strategy NE: Existence

Theorem

Every finite ordinal potential game has at least one pure strategy Nash equilibrium.

Intuition?

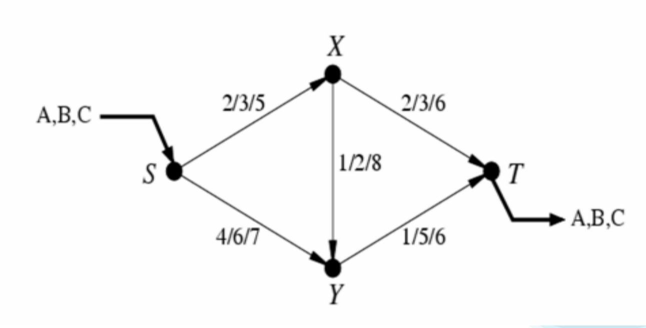
Does there exist a famous and representative potential game?

Yes! Congestion Game

But, you will see more surprising result soon!

Congestion Game: Example

- Three players from S to T
- a/b/c: cost when one/two/three players use that road
- Total cost of each player is the aggregate link cost over its path



Congestion Game: Formal Model

Congestion Model: $C = \langle \mathcal{I}, \mathcal{M}, (S_i)_{i \in \mathcal{I}}, (c^j)_{j \in \mathcal{M}} \rangle$ where:

- $\mathcal{I} = \{1, 2, \cdots, I\}$ is the set of players.
- $\mathcal{M} = \{1, 2, \cdots, m\}$ is the set of resources.
- S_i is the set of resource combinations (e.g., links or common resources) that player i can take/use. A strategy for player i is $s_i \in S_i$, corresponding to the subset of resources that this player is using.
- $c^j(k)$ is the benefit for the negative of the cost to each user who uses resource j if k users are using it.
- Define congestion game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ with utilities

$$u_i(s_i, s_{-i}) = \sum_{j \in s_i} c^j(k_j),$$

where k_j is the number of users of resource j under strategy s.

Congestion Game is a Potential Game

Theorem (Rosenthal (73))

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

Congestion game is not just an example

- Theorem
  - Each exact potential game has its equivalent congestion game.
- Hmm…

Potential game for continuous strategy set

- In the earlier slides,
  - Mainly, discrete, finite number of strategies
- Clearly, we can apply almost the similar principle to continuous strategy cases.
  - Example?
  - Conditions under which NE exists or even it is unqiue?
- How can we see some game is a potential game or not?
- We will see them later …

Summary

Pure Strategy NE: Existence

Theorem

Every finite ordinal potential game has at least one pure strategy Nash equilibrium.

- **Proof:** The global maximum of an ordinal potential function is a pure strategy Nash equilibrium. To see this, suppose that s^* corresponds to the global maximum. Then, for any $i \in \mathcal{I}$, we have, by definition, $\Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0$ for all $s \in S_i$. But since Φ is a potential function, for all i and all $s \in S_i$,

$$u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0 \quad \text{iff} \quad \Phi(s_i^*, s_{-i}^*) - \Phi(s, s_{-i}^*) \geq 0.$$

  Therefore, $u_i(s_i^*, s_{-i}^*) - u_i(s, s_{-i}^*) \geq 0$ for all $s \in S_i$ and for all $i \in \mathcal{I}$. Hence s^* is a pure strategy Nash equilibrium.
- Note, however, that there may also be other pure strategy Nash equilibria corresponding to local maxima.

Theorem (Rosenthal (73))

Every congestion game is a potential game and thus has a pure strategy Nash equilibrium.

- **Proof**: For each j define $\bar{k}_j^i$ as the usage of resource j excluding player i, i.e.,

$$\bar{k}_j^i = \sum_{i' \neq i} \mathbf{I}\left[j \in s_{i'}\right],$$

  where $\mathbf{I}\left[j \in s_{i'}\right]$ is the indicator for the event that $j \in s_{i'}$.
- With this notation, the utility difference of player i from two strategies s_i and s_i' (when others are using the strategy profile s_{-i}) is

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1).$$

- Now consider the function

$$\Phi(s) = \sum_{j \in \bigcup_{i' \in \mathcal{I}} s_{i'}} \left[\sum_{k=1}^{k_j} c^j(k)\right].$$

- We can also write

$$\Phi(s_i, s_{-i}) = \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k)\right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1).$$

- Therefore:

$$\begin{aligned}
\Phi(s_i, s_{-i}) - \Phi(s_i', s_{-i}) &= \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k)\right] + \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) \\
&\quad - \sum_{j \in \bigcup_{i' \neq i} s_{i'}} \left[\sum_{k=1}^{\bar{k}_j^i} c^j(k)\right] + \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= \sum_{j \in s_i} c^j(\bar{k}_j^i + 1) - \sum_{j \in s_i'} c^j(\bar{k}_j^i + 1) \\
&= u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}).
\end{aligned}$$