Lecture 4:
Continuous Normal-form game
and
Equilibrium efficiency and selection

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Key Words

- Normal-form (Strategic form) Game
- Matrix game
  - Strategy spaces are discrete
- Continuous-kernel game
  - Strategy spaces are continuous

- Strictly dominated strategies
- Pure/Mixed strategy
- Saddle point, Nash equilibrium

Continuous-kernel Game

- Action (strategy) sets have uncountably many elements
  - For example, strategies are:
  - Amount of transmission powers, access probabilities in Wi-Fi

- We will focus on pure strategies.

Example: Cournot Competition

- A famous example from microeconomics

> - Two firms producing a homogeneous good for the same market.
> - The action of a player i is a quantity, $s_i \in [0, \infty]$ (amount of good he produces).
> - The utility for each player is its total revenue minus its total cost,
>
> $$u_i(s_1, s_2) = s_i p(s_1 + s_2) - cs_i$$
>
> where $p(q)$ is the price of the good (as a function of the total amount q), and c is unit cost (same for both firms).
> - Assume for simplicity that $c = 1$ and $p(q) = \max\{0, 2 - q\}$

Recall: Best Response

Definition 9 *The* best response function $b_i(\mathbf{s}_{-i})$ *of a player i to the profile of strategies $\mathbf{s}_{-i}$ is a set of strategies for that player such that*

$$b_i(\mathbf{s}_{-i}) = \{s_i \in \mathcal{S}_i \mid u_i(s_i, \mathbf{s}_{-i}) \geq u_i(s'_i, \mathbf{s}_{-i}),\ \forall s'_i \in \mathcal{S}_i\}. \qquad (3.5)$$

Proposition 1 *A strategy profile $\mathbf{s}^* \in \mathcal{S}$ is a* Nash equilibrium *of a noncooperative game if and only if every player's strategy is a* best response *to the other players' strategies, that is:*

$$s_i^* \in b_i(\mathbf{s}^*_{-i}) \text{ for every player } i. \qquad (3.6)$$

Back to Cournot Competition

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$$u_i(s_1, s_2) = s_i p(s_1 + s_2) - cs_i$$

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Efficiency and Equilibrium Selection

Nash equilibrium: Efficiency

- Does the Nash equilibrium always exist?
- If so, are they "efficient"? Which is more "efficient"?
- Essentially, we need to compare two vectors

Pareto Optimality

- One measure of efficiency is Pareto optimality
  - A payoff vector **x** is Pareto optimal if there does **not** exist any payoff vector **y** such that

    $$\mathbf{y} \geq \mathbf{x}$$

    with at least one strict inequality for an element y_i

Example: Pareto Optimal

		Player 2: A	Player 2: B
Player 1	A	5, 5	1, 10
	B	10, 1	2, 2

Pareto Optimal (shaded cells: 5, 5; 1, 10; 10, 1)

- *Another type of solution concept: group rationality*
  - Pareto optimal

The Prisoner's Dilemma

- One of the most studied and used games
  - proposed in 1950s

		Player 2	
		A	B
Player 1	A	5, 5	1, 10
	B	10, 1	2, 2

better outcome

single NE

Price of Anarchy and Price of Stability

	S	C
S	5, 5	1, 10
C	10, 1	2, 2

- Price of Anarchy (PoA): (1+10)/(2+2)
  - Max aggregate payoff / min aggregate payoff at NE
- Price of Stability (PoS):
  - Max aggregate payoff/ max aggregate payoff at NE

What is PoA and PoS here?

- Two firms producing a homogeneous good for the same market.
- The action of a player i is a quantity, $s_i \in [0, \infty]$ (amount of good he produces).
- The utility for each player is its total revenue minus its total cost,

$$u_i(s_1, s_2) = s_i p(s_1 + s_2) - cs_i$$

  where $p(q)$ is the price of the good (as a function of the total amount q), and c is unit cost (same for both firms).
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Summary

Infinite Strategy Spaces

- Example: Cournot competition.
  - Two firms producing a homogeneous good for the same market.
  - The action of a player i is a quantity, $s_i \in [0, \infty]$ (amount of good he produces).
  - The utility for each player is its total revenue minus its total cost,

$$u_i(s_1, s_2) = s_i p(s_1 + s_2) - cs_i$$

    where $p(q)$ is the price of the good (as a function of the total amount q), and c is unit cost (same for both firms).
  - Assume for simplicity that $c = 1$ and $p(q) = \max\{0, 2 - q\}$
- Consider the best response correspondence for each of the firms, i.e., for each i, the mapping $B_i(s_{-i}) : S_{-i} \rightrightarrows S_i$ such that

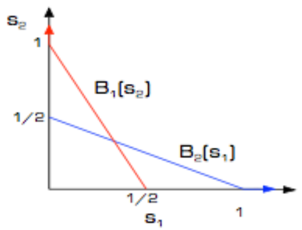
$$B_i(s_{-i}) \in \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Why is this a "correspondence" not a function? When will it be a function?

Cournot Competition (continued)

- By using the first order optimality conditions, we have

$$B_i(s_{-i}) = \arg\max_{s_i \geq 0}(s_i(2 - s_i - s_{-i}) - s_i) = \begin{cases} \frac{1-s_{-i}}{2} & \text{if } s_{-i} \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

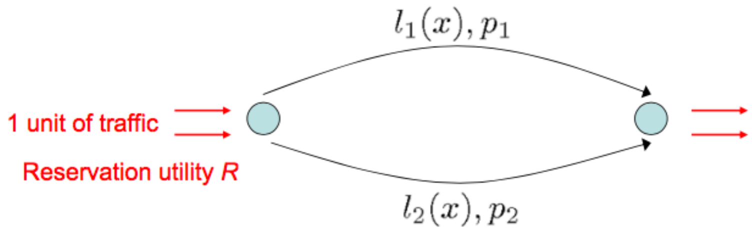


- The figure illustrates the best response correspondences (which in this case are functions).
- Assuming that players are **rational and fully knowledgeable about the structure of the game and each other's rationality**, what should the outcome of the game be?

Homework

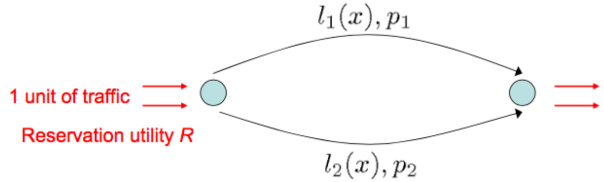
- Congestion pricing game
- 나중에 NE의 existence에서 써먹을 것임.

Congestion-Pricing Game (1)



- Consider a parallel link network with I links. Assume that d units of flow is to be routed through this network. We assume that this flow is the aggregate flow of many *infinitesimal* users.

Congestion-Pricing Game (2)



- Let $l_i(x_i)$ denote the latency function of link i, which represents the delay or congestion costs as a function of the total flow x_i on link i.
- Assume that the links are owned by independent providers. Provider i sets a price p_i per unit of flow on link i.
- The effective cost of using link i is $p_i + l_i(x_i)$.
- Users have a reservation utility equal to R, i.e., if $p_i + l_i(x_i) > R$, then no traffic will be routed on link i.

Example 1

- We consider an example with two links and latency functions $l_1(x_1) = 0$ and $l_2(x_2) = \frac{3x_2}{2}$. For simplicity, we assume that $R = 1$ and $d = 1$.
- Given the prices (p_1, p_2), we assume that the flow is allocated according to Wardrop equilibrium, i.e., the flows are routed along minimum effective cost paths and the effective cost cannot exceed the reservation utility.

Definition

A flow vector $x = [x_i]_{i=1,\dots,I}$ is a Wardrop equilibrium if $\sum_{i=1}^{I} x_i \le d$ and

$$p_i + l_i(x_i) = \min_j \{p_j + l_j(x_j)\}, \quad \text{for all } i \text{ with } x_i > 0,$$

$$p_i + l_i(x_i) \le R, \quad \text{for all } i \text{ with } x_i > 0,$$

with $\sum_{i=1}^{I} x_i = d$ if $\min_j\{p_j + l_j(x_j)\} < R$.

- We use the preceding characterization to determine the flow allocation on each link given prices $0 \le p_1, p_2 \le 1$:

$$x_2(p_1, p_2) = \begin{cases} \frac{2}{3}(p_1 - p_2), & p_1 \ge p_2, \\ 0, & \text{otherwise}, \end{cases}$$

  and $x_1(p_1, p_2) = 1 - x_2(p_1, p_2)$.
- The payoffs for the providers are given by:

$$u_1(p_1, p_2) = p_1 \times x_1(p_1, p_2)$$
$$u_2(p_1, p_2) = p_2 \times x_2(p_1, p_2)$$

- We find the pure strategy Nash equilibria of this game by characterizing the best response correspondences, $B_i(p_{-i})$ for each player.
  - The following analysis assumes that at the Nash equilibria (p_1, p_2) of the game, the corresponding Wardrop equilibria x satisfies $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 = 1$. For the proofs of these statements, see [Acemoglu and Ozdaglar 07].

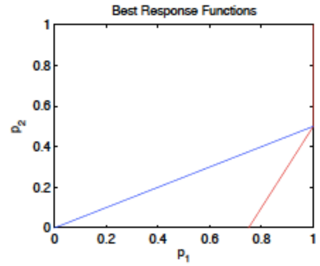
- In particular, for a given p_2, $B_1(p_2)$ is the optimal solution set of the following optimization problem

$$\begin{aligned} \text{maximize}_{0\leq p_1\leq 1,\ 0\leq x_1\leq 1} \quad & p_1 x_1 \\ \text{subject to} \quad & p_1 = p_2 + \frac{3}{2}(1-x_1) \end{aligned}$$

- Solving the preceding optimization problem, we find that

$$B_1(p_2) = \min\left\{1, \frac{3}{4} + \frac{p_2}{2}\right\}.$$

Similarly, $B_2(p_1) = \frac{p_1}{2}$.



- The figure illustrates the best response correspondences as a function of p_1 and p_2. The correspondences intersect at the unique point $(p_1, p_2) = (1, \frac{1}{2})$, which is the unique pure strategy equilibrium.

Example 2

- We next consider a similar example with latency functions given by

$$l_1(x) = 0, \qquad l_2(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ \frac{x-1/2}{\epsilon} & x \geq 1/2, \end{cases}$$

for some sufficiently small $\epsilon > 0$.

- The following list considers all candidate Nash equilibria (p_1, p_2) and profitable unilateral deviations for ϵ sufficiently small, thus establishing the nonexistence of a pure strategy Nash equilibrium:
  - $p_1 = p_2 = 0$: A small increase in the price of provider 1 will generate positive profits, thus provider 1 has an incentive to deviate.
  - $p_1 = p_2 > 0$: Let x be the corresponding flow allocation. If $x_1 = 1$, then provider 2 has an incentive to decrease its price. If $x_1 < 1$, then provider 1 has an incentive to decrease its price.
  - $0 \leq p_1 < p_2$: Player 1 has an incentive to increase its price since its flow allocation remains the same.
  - $0 \leq p_2 < p_1$: For ϵ sufficiently small, the profit function of player 2, given p_1, is strictly increasing as a function of p_2, showing that provider 2 has an incentive to increase its price.