Lecture 2:
Normal-form game
(Strategic-form game)
with pure strategies

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Key Words

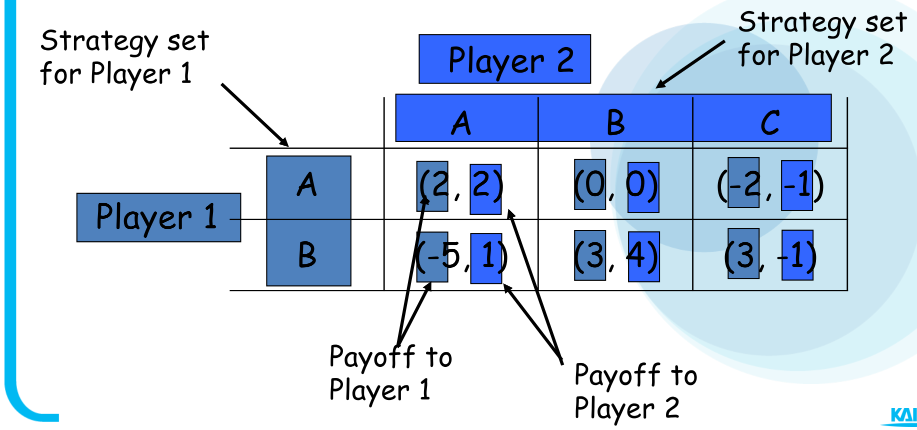
- Normal-form (Strategic form) Game
- Matrix game
  - Strategy spaces are discrete
- Continuous-kernel game
  - Strategy spaces are continuous

- Strictly dominated strategies
- Pure/Mixed strategy
- Saddle point, Nash equilibrium

Matrix Game:
Pure Strategy

Matrix Game

- Representation of a game
- Simultaneous play
  - players analyze the game and write their strategy on a paper
- Combination of strategies determines payoff



	A	B	C
A	(2, 2)	(0, 0)	(-2, -1)
B	(-5, 1)	(3, 4)	(3, -1)

More Formal Game Definition

- Normal form (strategic) game
  - a finite set N of players
  - a set strategies A_i for each player i
  - payoff function $u_i(s)$ for each player $i \in N$
    - where $s \in A = \times_{j \in N} A_j$ is the set of strategies chosen by all players
      $i \in N$
- A is the set of all possible outcomes
- $s \in A$ is a set of strategies chosen by players
  - defines an outcome
- $u_i : A \rightarrow \Re$

Two-person Zero-sum Games

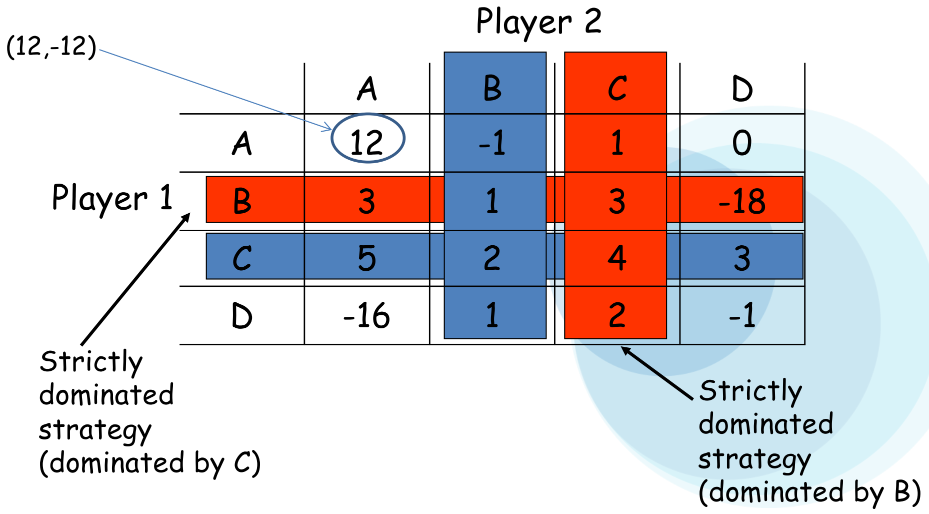
- One of the first games studied
  - most well understood type of game
- Players interest are strictly opposed
  - what one player gains what the other loses
  - game matrix has single entry (gain to player 1)
- Intuitive solution concept
  - players maximize gains
  - unique solution

Solution Concept

- A formal rule for predicting how a game will be played
- Describes which strategies will be adopted by palyers, and thus the result of the game
- Many kinds of solution concepts
  - People's perspectives are different
- It does not talk about how players reach a solution concept
- Thus, naturally, it is an "equilibrium concept".

Analyzing the Game: Domination

- Player 1 maximizes matrix entry, while player 2 minimizes



Player 2

Player 1

	A	B	C	D
A	12	-1	1	0
B	3	1	3	-18
C	5	2	4	3
D	-16	1	2	-1

(12,-12)

Strictly dominated strategy (dominated by C)

Strictly dominated strategy (dominated by B)

Dominance

- Strategy S *strictly dominates* a strategy T
  - if every possible outcome when S is chosen is better than the corresponding outcome when T is chosen
- Dominance Principle
  - rational players never choose strictly dominated strategies
- *Idea:* Solve the game by eliminating strictly dominated strategies!
  - iterated removal

Solving the Game

- Iterated removal of strictly dominated strategies

Player 2

Player 1		~~L~~	M	~~R~~
	~~T~~	~~2~~	~~1~~	~~1~~
	B	~~3~~	2	~~3~~

- Player 1 cannot remove any strategy (neither T or B dominates the other)
- Player 2 can remove strategy R (dominated by M)
- Player 1 can remove strategy T (dominated by B)
- Player 2 can remove strategy L (dominated by M)
- Solution: P_1 -> B, P_2 -> M
  - payoff of 2

Solving the Game

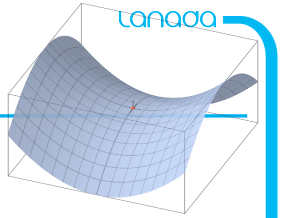
- Removal of strictly dominates strategies does not always work
- Consider the game

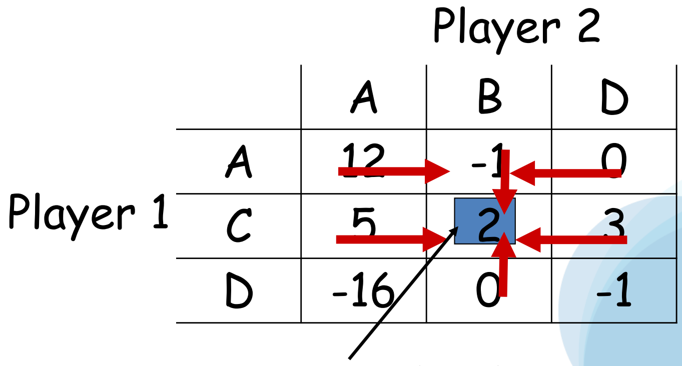
Player 2

Player 1		A	B	D
	A	12	-1	0
	C	5	2	3
	D	-16	0	-1

- Neither player has dominated strategies
- Requires another solution concept

Analyzing the Game





Player 2

Player 1		A	B	D
	A	12	-1	0
	C	5	2	3
	D	-16	0	-1

Outcome (C, B) seems "stable"

- saddle point of game

Saddle Points

- An outcome is a *saddle point*
  - if it is both less than or equal to any value in its row and greater than or equal to any value in its column
- Saddle Point Principle
  - Players should choose outcomes that are saddle points of the game
- Value of the game
  - value of saddle point outcome if it exists

Why Play Saddle Points?

		Player 2		
		A	B	D
Player 1	A	12	-1	0
	C	5	2	3
	D	-16	0	-1

- If player 1 believes player 2 will play B
  - player 1 should play best response to B (which is C)
- If player 2 believes player 1 will play C
  - player 2 should play best response to C (which is B)

Why Play Saddle Points?

		Player 2		
		A	B	D
Player 1	A	12	-1	0
	C	5	2	3
	D	-16	0	-1

- Why should player 1 believe player 2 will play B?
  - playing B guarantees player 2 *loses at most v* (which is 2)
- Why should player 2 believe player 1 will play C?
  - playing C guarantees player 1 *wins at least v* (which is 2)

Powerful arguments to play saddle point!

Solving the Game (min-max algorithm)

		Player 2				
		A	B	C	D	
Player 1	A	4	3	2	5	2
	B	-10	2	0	-1	-10
	C	7	5	1	3	1
	D	0	8	-4	-5	-5
		7	8	2	5	

- choose maximum entry in each column
- choose the minimum among these
- this is the minimax value

- choose minimum entry in each row
- choose the maximum among these
- this is maximin value

- if minimax == maximin, then this is the saddle point of game

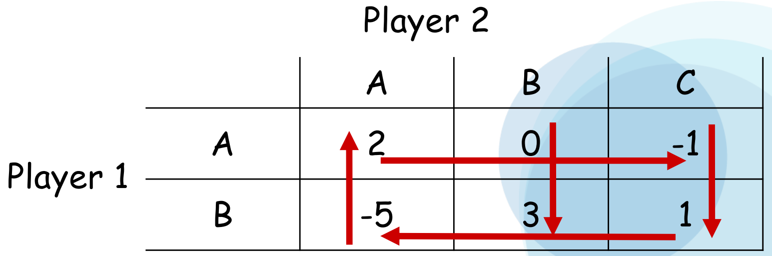
Multiple Saddle Points

- In general, game can have multiple saddle points

Player 2

Player 1	A	B	C	D	
A	3	2	2	5	②
B	2	-10	0	-1	-10
C	5	2	2	3	②
D	8	0	-4	-5	-5
	8	②	②	5	

Games With no Saddle Points



Player 2

Player 1	A	B	C
A	2	0	-1
B	-5	3	1

Two-person Non-zero Sum Games

- Players are not strictly opposed
  - payoff sum is non-zero

Player 2

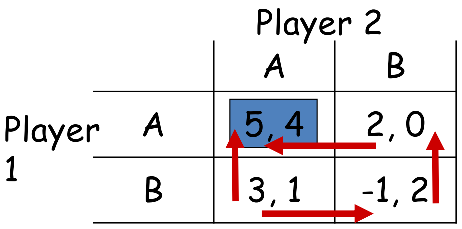
Player 1	A	B
A	3, 4	2, 0
B	5, 1	-1, 2

- Situations where interest is not directly opposed

What is the Solution?

- Ideas of zero-sum game: saddle points

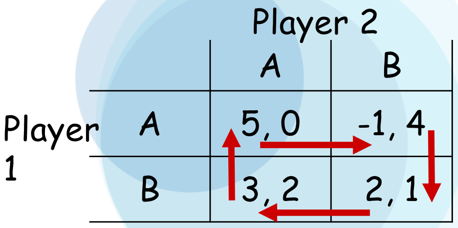
- pure strategy equilibrium



Player 2

Player 1	A	B
A	5, 4	2, 0
B	3, 1	-1, 2

- no pure strategy eq.



Player 2

Player 1	A	B
A	5, 0	-1, 4
B	3, 2	2, 1

Nash equilibrium

- A **Nash equilibrium** is a strategy profile **s*** with the property that no player *i* can do better by choosing a strategy different from s*, given that every other player *j* ≠ *i* .
- In other words, for each player *i* with payoff function u_i ,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \forall s_i \in \mathcal{S}_i$$

- No user can change its payoff by **unilaterally** changing its strategy, i.e., changing its strategy while $\mathbf{s}_{-i}$ is fixed

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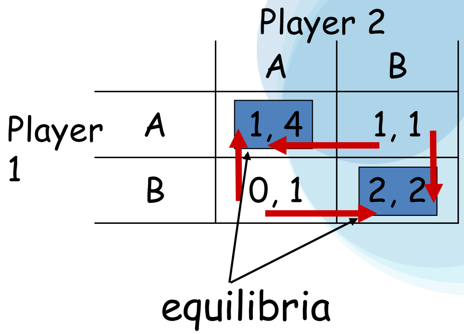
Multiple Solution Problem

- Games can have multiple equilibria
  - not equivalent:
    - payoff is different
  - not interchangeable:
    - playing an equilibrium strategy does not lead to equilibrium



		Player 2	
		A	B
Player 1	A	1, 4	1, 1
	B	0, 1	2, 2

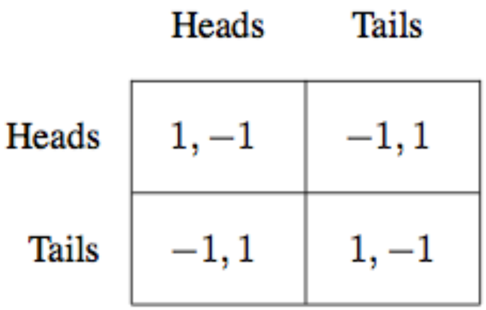
Ex 1: Coordination game

- Two drivers, driving towards each other

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Ex 2: Matching Pennies game

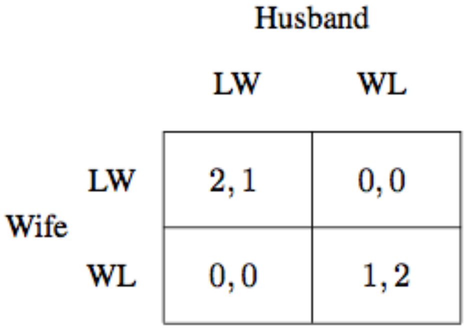
- Each player shows her coin.
- Same side →Player 1 pockets both, and Player 2 does otherwise.



	Heads	Tails
Heads	$1,-1$	$-1,1$
Tails	$-1,1$	$1,-1$

Ex 3: Battle of the Sexes Game

- Tries to see a movie
- Husband: “Lethal Weapon”, Wife: “Wondrous Love”



		Husband	
		LW	WL
Wife	LW	2, 1	0, 0
	WL	0, 0	1, 2

Summary