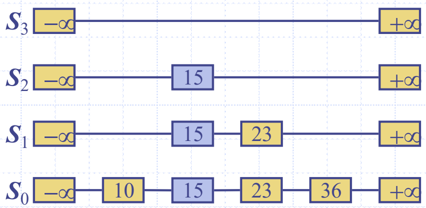
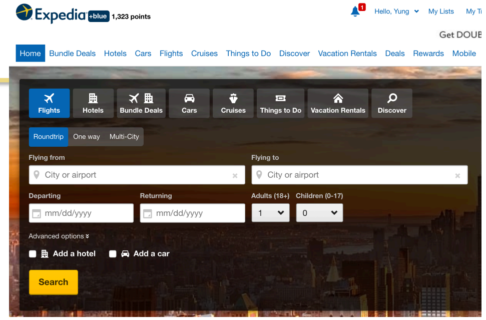
Skip Lists



Flight Databases



- Key: (origin, destination, date, time)
- Value: (flight number, available seats, first or economy, duration, fare, etc
- User
  - Happy about "closest" departure time, not simply searching the flights with exact match with the given key
- We may need a slightly different data structure from Map ADT
- That's an ordered Map

Ordered Map ADT

- Map ADT + the following methods

firstEntry(k): Return an iterator to the entry with smallest key value; if the map is empty, it returns end.

lastEntry(k): Return an iterator to the entry with largest key value; if the map is empty, it returns end.

ceilingEntry(k): Return an iterator to the entry with the least key value greater than or equal to k; if there is no such entry, it returns end.

floorEntry(k): Return an iterator to the entry with the greatest key value less than or equal to k; if there is no such entry, it returns end.

lowerEntry(k): Return an iterator to the entry with the greatest key value less than k; if there is no such entry, it returns end.

higherEntry(k): Return an iterator to the entry with the least key value greater than k; if there is no such entry, it returns end.

- SkipList is one of the efficient way of implementing ordered Maps

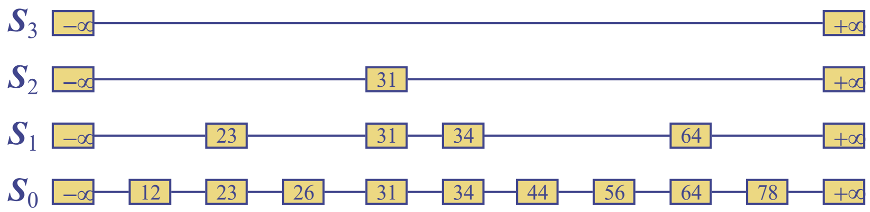
Implementing Ordered Map

- Natural choice
  - Sorted list based implementation
  - O(n) searching, insertion, deletion complexity
- Lesson from HashTable
  - Unordered map can be implemented by O(1) time (on average)
- Can we imagine similar things for ordered map?
  - SkipList

Operation	***Time***
size, empty	$O(1)$
firstEntry, lastEntry	$O(1)$
find, insert, erase	$O(\log n)$ (expected)
ceilingEntry, floorEntry, lowerEntry, higherEntry	$O(\log n)$ (expected)

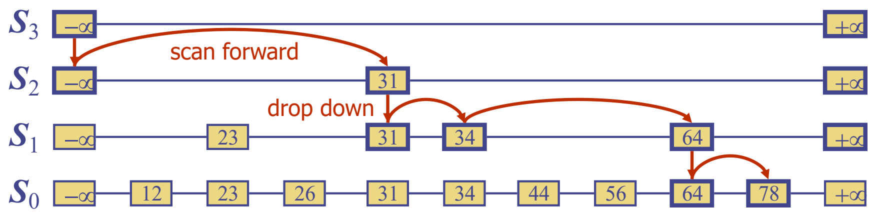
What is a Skip List

- A skip list for a set S of distinct (key, element) items is a series of lists $S_0, S_1, \ldots, S_h$ such that
  - Each list S_i contains the special keys $+\infty$ and $-\infty$
  - List S_0 contains the keys of S in nondecreasing order
  - Each list is a subsequence of the previous one, i.e.,
    $$S_0 \supseteq S_1 \supseteq \ldots \supseteq S_h$$
  - List S_h contains only the two special keys
- We show how to use a skip list to implement the ordered MAP ADT



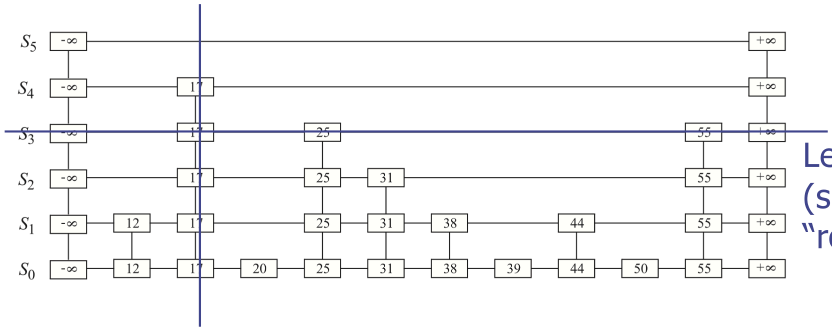
Search

- We search for a key x in a a skip list as follows:
  - We start at the first position of the top list
  - At the current position p, we compare x with $y \leftarrow key(next(p))$
    - $x = y$: we return $element(next(p))$
    - $x > y$: we "scan forward"
    - $x < y$: we "drop down"
  - If we try to drop down past the bottom list, we return ***null***
- Example: search for 78



Terminology

Height = 5



Level (similar to "row")

Tower (similar to "column")

(Note) Randomized Algorithms

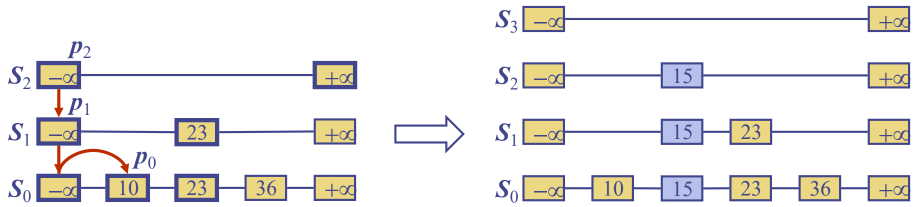
- A randomized algorithm performs coin tosses (i.e., uses random bits) to control its execution
- It contains statements of the type

  $b \leftarrow random()$
  if $b = 0$
  do A …
  else $\{ b = 1 \}$
  do B …

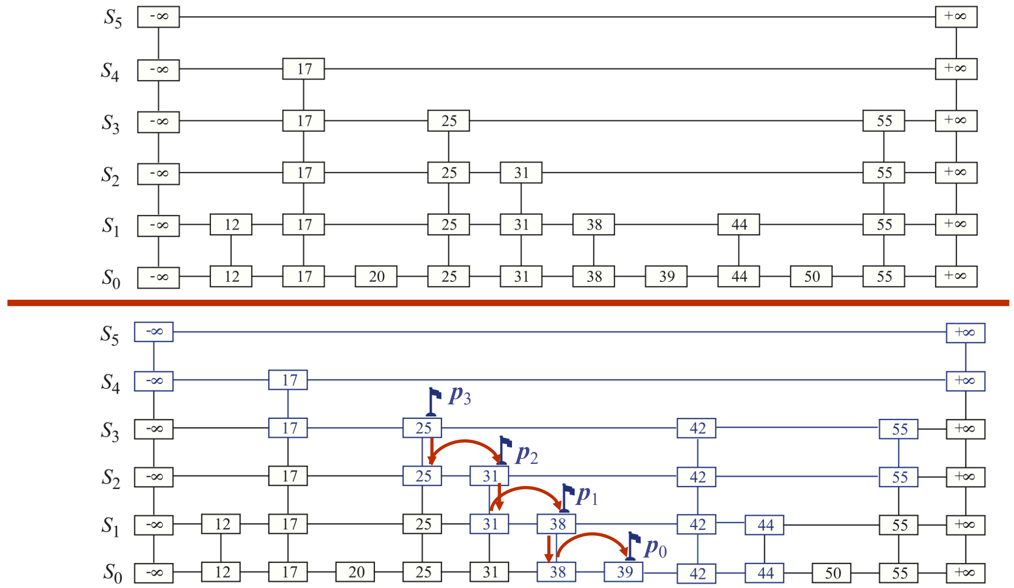
- Its running time depends on the outcomes of the coin tosses
- We analyze the expected running time of a randomized algorithm under the following assumptions
  - the coins are unbiased, and
  - the coin tosses are independent
- The worst-case running time of a randomized algorithm is often large but has very low probability (e.g., it occurs when all the coin tosses give "heads")
- We use a randomized algorithm to insert items into a skip list

Insertion

- To insert an entry (x, o) into a skip list, we use a randomized algorithm:
  - We repeatedly toss a coin until we get tails, and we denote with i the number of times the coin came up heads
  - If $i \geq h$, we add (to the skip list) new lists $S_{h+1}, \ldots, S_{i+1}$, each containing only the two special keys, and do nothing, otherwise
  - We search for x in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with largest key less than x in each list $S_0, S_1, \ldots, S_i$
  - For $j \leftarrow 0, \ldots, i$, we insert item (x, o) into list S_j after position p_j
- Example: insert key 15, with $i = 2$

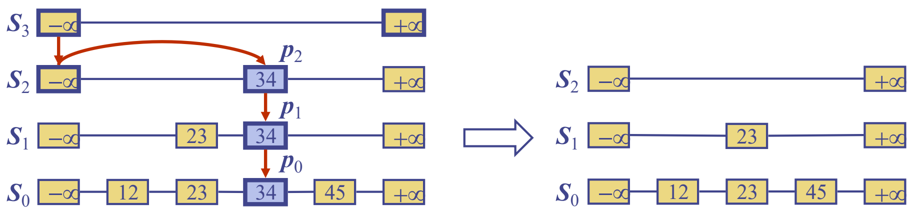


Example of Insertion of Key "42" with i = 3



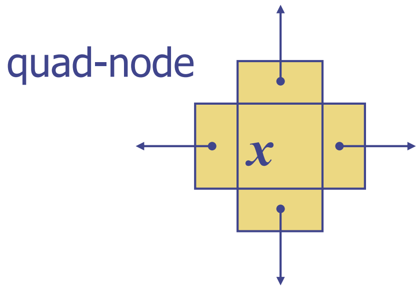
Deletion

- To remove an entry with key x from a skip list, we proceed:
  - We search for x in the skip list and find the positions $p_0, p_1, \ldots, p_i$ of the items with key x, where position p_j is in list S_j
  - We remove positions $p_0, p_1, \ldots, p_i$ from the lists $S_0, S_1, \ldots, S_i$
  - We remove all but one list containing only the two special keys
- Example: remove key 34



Implementation

- We can implement a skip list with quad-nodes
- A quad-node stores:
  - entry
  - link to the node prev
  - link to the node next
  - link to the node below
  - link to the node above
- Also, we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them



Performance of skiplist

- Space
  - O(n)
  - Surprising? (Note that an element is stored in multiple places)
- Time

Operation	*Time*
size, empty	$O(1)$
firstEntry, lastEntry	$O(1)$
find, insert, erase	$O(\log n)$ (expected)
ceilingEntry, floorEntry, lowerEntry, higherEntry	$O(\log n)$ (expected)

- You will see why the probability course helps here. Be ready for math

Space Usage

- The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm
- We use the following two basic probabilistic facts:

  Fact 1: The probability of getting i consecutive heads when flipping a coin is $1/2^i$

  Fact 2: If each of n entries is present in a set with probability p, the expected size of the set is np ***(expectation of a binomial distribution)***

- Consider a skip list with n entries
  - By Fact 1, we insert an entry in list S_i with probability $1/2^i$
    - Why? Because we insert the entry for all levels <= i
  - By Fact 2, the expected size of list S_i is $n/2^i$
- The expected number of nodes used by the skip list is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i} < 2n$$

- Thus, the expected space usage of a skip list with n items is $O(n)$

Height

- The running time of the search and insertion algorithms is affected by the height h of the skip list
- We show that with high probability, a skip list with n items has height $O(\log n)$
- We use the following additional probabilistic fact:

  Fact 3: If each of n events has probability p, the probability that at least one event occurs is at most np ***(union bound)***

- Consider a skip list with n entires
  - By Fact 1, we insert an entry in list S_i with probability $1/2^i$
  - By Fact 3, the probability that list S_i has at least one item is at most $n/2^i$
- By picking $i = 3\log n$, we have that the probability that $S_{3\log n}$ has at least one entry is at most
  $n/2^{3\log n} = n/n^3 = 1/n^2$
- Thus a skip list with n entries has height at most $3\log n$ with probability at least $1 - 1/n^2$
- The height is *O(log* n*)* with "high probability"

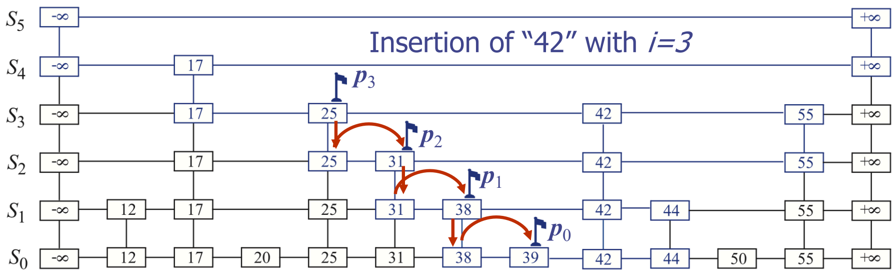
Search and Update Times (1)

- The search time in a skip list is proportional to
  - the number of drop-down steps, plus
  - the number of scan-forward steps
- The drop-down steps are bounded by the height of the skip list and thus are $O(\log n)$ with high probability
- To analyze the scan-forward steps, we use yet another probabilistic fact:

  Fact 4: The expected number of coin tosses required in order to get tails is 2

Search and Update Times (2)

- After the key at the starting position, each additional key examined in a scan-forward at level *i* cannot also belong to level *i+1*
  - *The probability that any further key is examined is ½*
  - *How many additional keys should be examined at each level i on average?*
- By Fact 4, in each list the expected number of scan-forward steps is 2, i.e., $O(1)$
- Thus, the expected number of scan-forward steps is $O(\log n)$
- We conclude that a search in a skip list takes $O(\log n)$ expected time
- The analysis of insertion and deletion gives similar results



Summary

- A skip list is a data structure for dictionaries that uses a randomized insertion algorithm
- In a skip list with n entries
  - The expected space used is $O(n)$
  - The expected search, insertion and deletion time is $O(\log n)$
- Using a more complex probabilistic analysis, one can show that these performance bounds also hold with high probability
- Skip lists are fast and simple to implement in practice

Questions?