Priority Queues

Introduction

- Priority Queue
  - Data structure for storing a collection of prioritized elements
  - Supporting arbitrary element insertion
  - Supporting removal of elements in order of priority

- So far, we covered "position-based" data structures
  - Stacks, queues, deques, lists, and even lists
  - Store elements at specific positions (linear or hierarchical)
  - Insertion and removal based on "position" (linear or hierarchical)
  - But, priority queue
    - Insertion and removal: priority-based

- Question: how to express the priority of an element
  - Key (example: your student id)

Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a pair (key, value), where the key indicates the priority
- Main methods of the Priority Queue ADT
  - insert(e)
    inserts an entry e (with an implicit associated key value)
  - removeMin()
    removes the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), empty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Total Order Relations (a topic of Discrete Math)

- Keys in a priority queue can be arbitrary objects on which an order is defined
- Two distinct entries in a priority queue can have the same key
- Total ordering
  - Comparison rule should be defined for every pair of keys

- Mathematical concept of total order relation $\leq$
  - Reflexive property: $x \leq x$
  - Antisymmetric property: $x \leq y \wedge y \leq x \Rightarrow x = y$
  - Transitive property: $x \leq y \wedge y \leq z \Rightarrow x \leq z$
- Satisfying the above three properties ensures:
  - Never leading to a comparison contradiction

Example: Total order & Partial order

- 2D points with (x-coordinate, y-coordinate)
  - Define relation '>=' based on x-first, and y-next
  - (4,3) >= (3,4), (3,5) >= (3,4)
  - Total ordering

  - What about defining relation '>=' based on both x and y
  - (4,3) >=(2,1), but (4,3) ??? (3,4)
  - Partial ordering
    - Comparison not defined for some objects

- We assume that we define a comparison that leads to total ordering.

Priority Queue Sorting

- We can use a priority queue to sort a set of comparable elements
  1. Insert the elements one by one with a series of insert operations
  2. Remove the elements in sorted order with a series of removeMin operations
- The running time of this sorting method depends on the priority queue implementation

Algorithm ***PQ-Sort*(*S*, *C*)**
    Input sequence ***S***, comparator ***C*** for the elements of ***S***
    Output sequence ***S*** sorted in increasing order according to ***C***
    P ← priority queue with comparator ***C***
    while ¬***S.empty*** ()
        e ← ***S.front***(); ***S.eraseFront***()
        P.insert (***e***, ∅)
    while ¬***P.empty***()
        e ← ***P.removeMin***()
        S.insertBack(***e***)

Sequence-based Priority Queue

- Implementation with an unsorted list

- Performance:
  - insert takes $O(1)$ time since we can insert the item at the beginning or end of the sequence
  - removeMin and min take $O(n)$ time since we have to traverse the entire sequence to find the smallest key

- Implementation with a sorted list

- Performance:
  - insert takes $O(n)$ time since we have to find the place where to insert the item
  - removeMin and min take $O(1)$ time, since the smallest key is at the beginning

Selection-Sort

- Selection-sort is the variation of PQ-sort where the priority queue is implemented with an unsorted sequence
- Running time of Selection-sort:
  1. Inserting the elements into the priority queue with n insert operations takes $O(n)$ time
  2. Removing the elements in sorted order from the priority queue with n removeMin operations takes time proportional to

$$1 + 2 + \ldots + n$$

- Selection-sort runs in $O(n^2)$ time

Selection-Sort Example

	Sequence S	Priority Queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(7,4)
..	.. ..	
(g)	()	(7,4,8,2,5,3,9)
Phase 2		
(a)	(2)	(7,4,8,5,3,9)
(b)	(2,3)	(7,4,8,5,9)
(c)	(2,3,4)	(7,8,5,9)
(d)	(2,3,4,5)	(7,8,9)
(e)	(2,3,4,5,7)	(8,9)
(f)	(2,3,4,5,7,8)	(9)
(g)	(2,3,4,5,7,8,9)	()

Insertion-Sort

- Insertion-sort is the variation of PQ-sort where the priority queue is implemented with a sorted sequence
- Running time of Insertion-sort:
  1. Inserting the elements into the priority queue with n insert operations takes time proportional to

     $$1 + 2 + \ldots + n$$

  2. Removing the elements in sorted order from the priority queue with a series of n removeMin operations takes $O(n)$ time
- Insertion-sort runs in $O(n^2)$ time

Insertion-Sort Example

	Sequence S	Priority queue P
Input:	(7,4,8,2,5,3,9)	()
Phase 1		
(a)	(4,8,2,5,3,9)	(7)
(b)	(8,2,5,3,9)	(4,7)
(c)	(2,5,3,9)	(4,7,8)
(d)	(5,3,9)	(2,4,7,8)
(e)	(3,9)	(2,4,5,7,8)
(f)	(9)	(2,3,4,5,7,8)
(g)	()	(2,3,4,5,7,8,9)
Phase 2		
(a)	(2)	(3,4,5,7,8,9)
(b)	(2,3)	(4,5,7,8,9)
..	..	..
(g)	(2,3,4,5,7,8,9)	()

Comparator

Another design method

How to define order for any object? (1)

- Integer, float, double
  - Quite clear on how to define "order"
- Student: id, sex, department
  - S1 is less than S2? In what sense?
- Flight Passengers: airplane number, seat number, sex
  - P1 is less than P2? In what sense?
- How to design "comparison logic" in a programming language?
- What design is good?

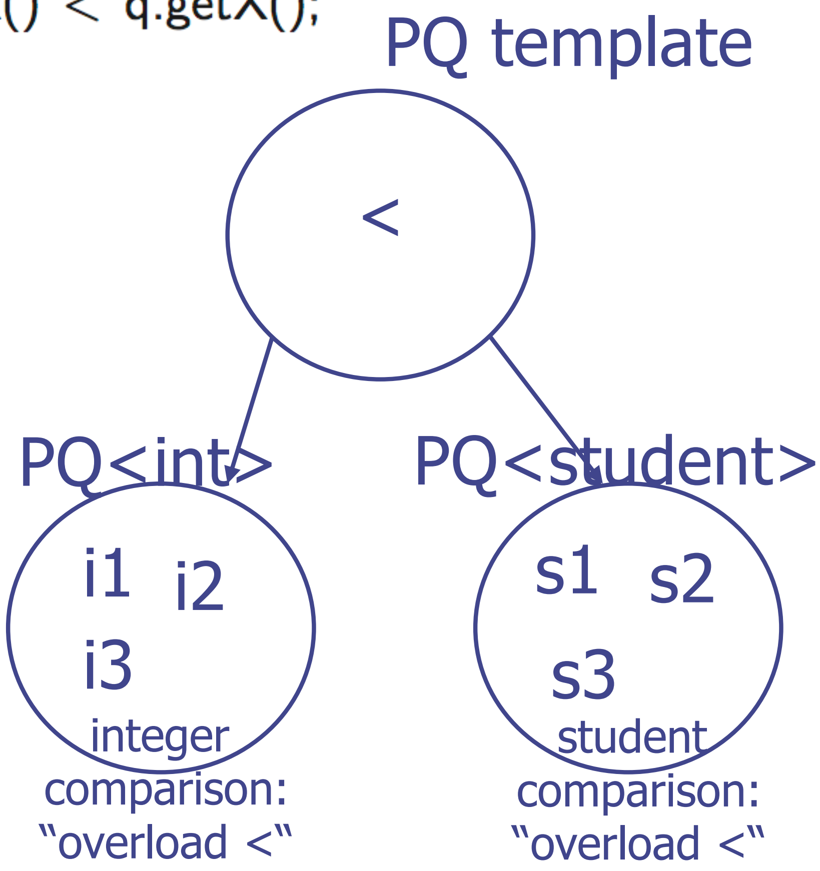
Design 1: Separate Design

- Different Priority Queue based on the element type and the manner of comparing elements
- PQ_Int, PQ_Student, PQ_XXX
- Simple, but not general
- Many copies of the same code

Design 2: Template and Overloading (2)

```
bool operator<(const Point2D& p, const Point2D& q) {
  if (p.getX() == q.getX())    return p.getY() < q.getY();
  else                         return p.getX() < q.getX();
}
```

- General enough for many situations
- But,
  - Cannot have multiple comparison methods for the same type
  - What about comparison based on y-first, and x-next?
- Even for the same data type, we want to apply different comparison methods A or B, depending on the situations



Design 3: Separating Comparator (1)

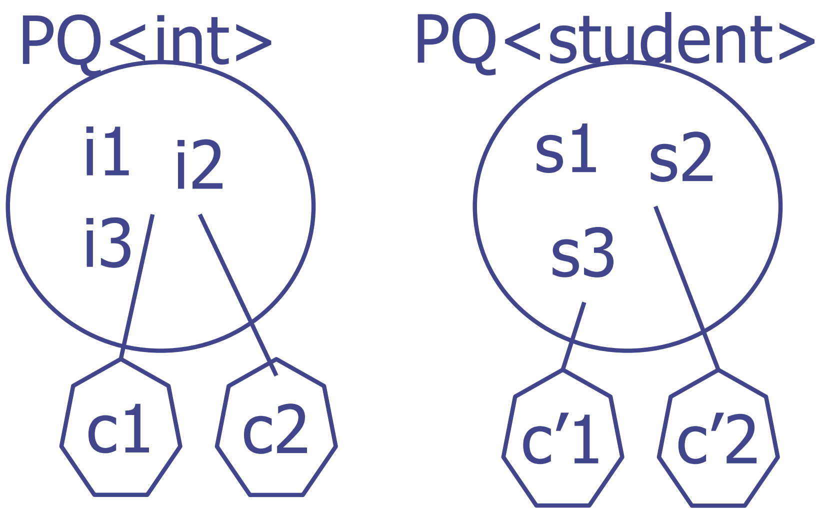
- 2D points: Point2D p, Point 2: q
  - Sometimes we want either of
    X-based comparison, Y-based comparison
- Idea
  - Define a comparator class, e.g., “LeftRight” (x-based) and “BottomTop” (y-based)
  - Overload “()” operator

```
class LeftRight {                              // a left-right comparator
public:
  bool operator()(const Point2D& p, const Point2D& q) const
    { return p.getX() < q.getX(); }
};

class BottomTop {                              // a bottom-top comparator
public:
  bool operator()(const Point2D& p, const Point2D& q) const
    { return p.getY() < q.getY(); }
};
```

Design 3: Separating Comparator (2)

```
Point2D p(1.3, 5.7), q(2.5, 0.6);          // two points
LeftRight leftRight;                       // a left-right comparator
BottomTop bottomTop;                       // a bottom-top comparator
printSmaller(p, q, leftRight);             // outputs: (1.3, 5.7)
printSmaller(p, q, bottomTop);             // outputs: (2.5, 0.6)
```

```
template <typename E, typename C>        // element type and comparator
void printSmaller(const E& p, const E& q, const C& isLess) {
  cout << (isLess(p, q) ? p : q) << endl;  // print the smaller of p and q
}
```

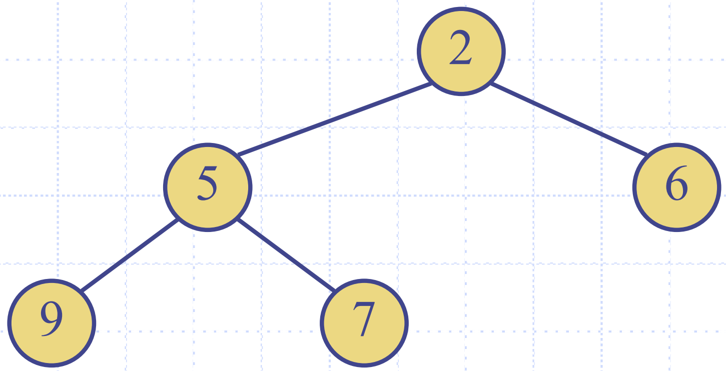


In C++

```
#include <algorithm>
#include <functional>
#include <array>
#include <iostream>

// sort using a custom function object
struct MyLess{
  bool operator()(int a, int b) const
  {       return a > b;  }
};


int main()
{
    std::array<int, 10> s = {5, 7, 4, 2, 8, 6, 1, 9, 0, 3};

    // sort using the default operator<
    std::sort(s.begin(), s.end());
    for (int i=0 ; i<s.size();i++) {
        std::cout << s[i] << " ";
    }
    std::cout << '\n';

    MyLess myless;

    std::sort(s.begin(), s.end(), myless);

    for (int i=0 ; i<s.size();i++) {
        std::cout << s[i] << " ";
    }
    std::cout << '\n';

}
```

```
[yi@iMacyung ~/tmp]# ./a.out
0 1 2 3 4 5 6 7 8 9
9 8 7 6 5 4 3 2 1 0
```

Heaps



Recall Priority Queue ADT

- A priority queue stores a collection of entries
- Typically, an entry is a pair (key, value), where the key indicates the priority
- Main methods of the Priority Queue ADT
  - insert(e) inserts an entry e
  - removeMin()
    removes the entry with smallest key
- Additional methods
  - min()
    returns, but does not remove, an entry with smallest key
  - size(), empty()
- Applications:
  - Standby flyers
  - Auctions
  - Stock market

Recall PQ Sorting

- We use a priority queue
  - Insert the elements with a series of insert operations
  - Remove the elements in sorted order with a series of removeMin operations
- The running time depends on the priority queue implementation:
  - Unsorted sequence gives selection-sort: $O(n^2)$ time
  - Sorted sequence gives insertion-sort: $O(n^2)$ time
- Can we do better? Balancing the above

Algorithm *PQ-Sort*(*S*, *C*)

Input sequence *S*, comparator *C* for the elements of *S*

Output sequence *S* sorted in increasing order according to *C*

P ← priority queue with comparator *C*

while ¬*S.empty* ()

e ← *S.front*(); *S.eraseFront*()

P.insert (*e*, ∅)

while ¬*P.empty*()

e ← *P.removeMin*()

S.insertBack(*e*)

We will have these results soon …

Sequence-based

Operation	***Unsorted List***	***Sorted List***
size, empty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

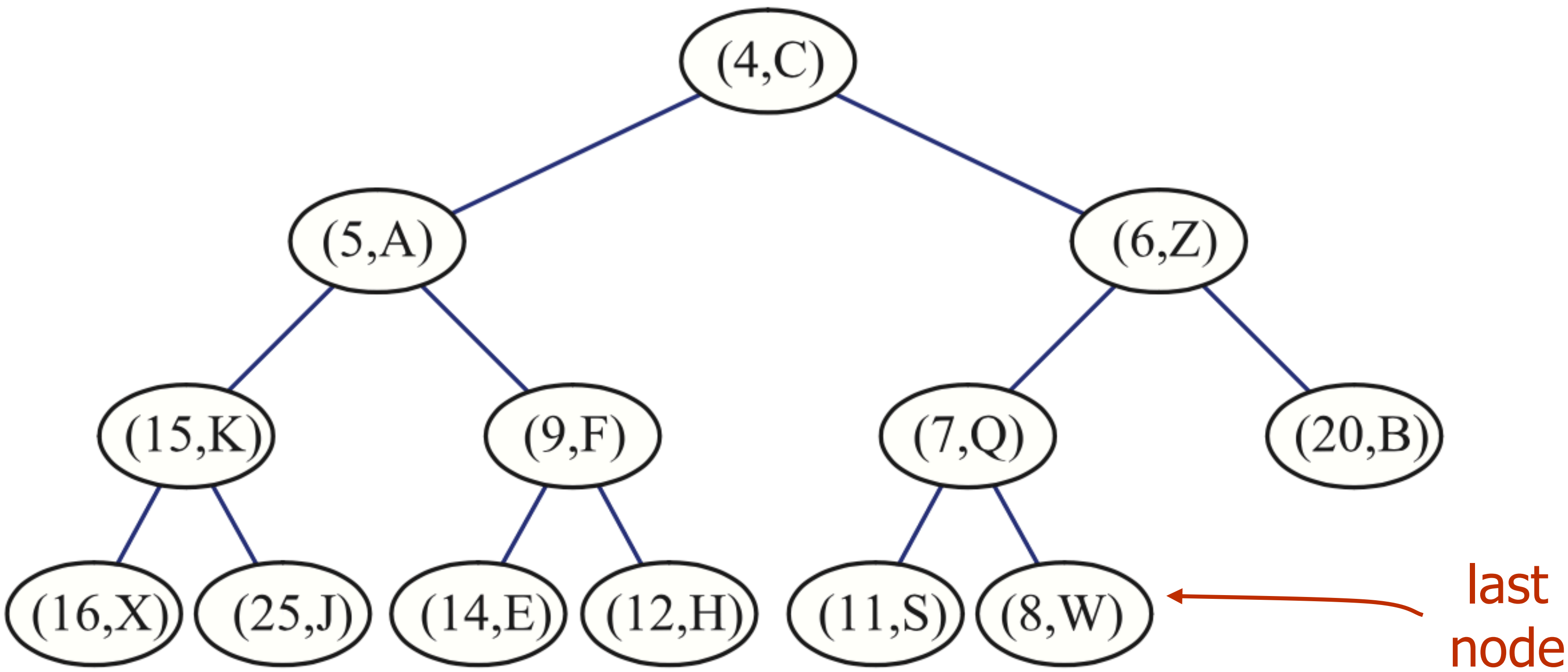
Heap-based

Operation	***Time***
size, empty	$O(1)$
min	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$

Key: Where were the “unnecessary repetitions” and “stupidity”?

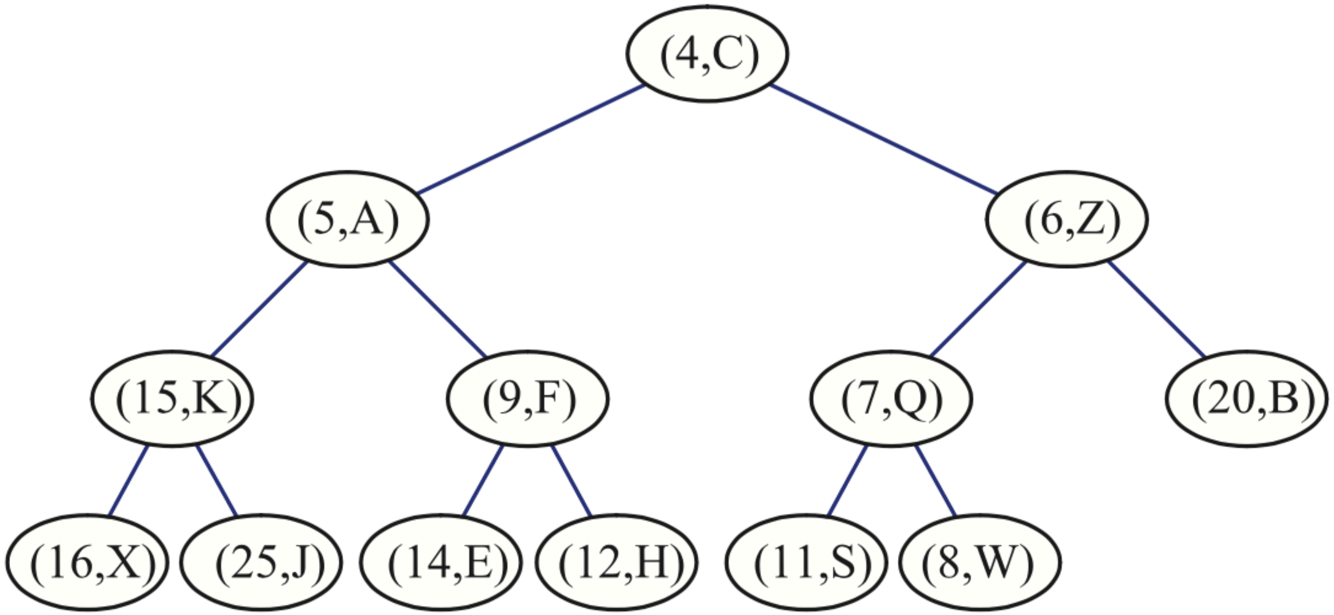
Heap: Overview

- A heap is a binary tree storing keys at its nodes and satisfying the following properties:
  - 1. Heap-order property
  - 2. Complete binary tree property
- The last node of a heap is the rightmost node of maximum depth



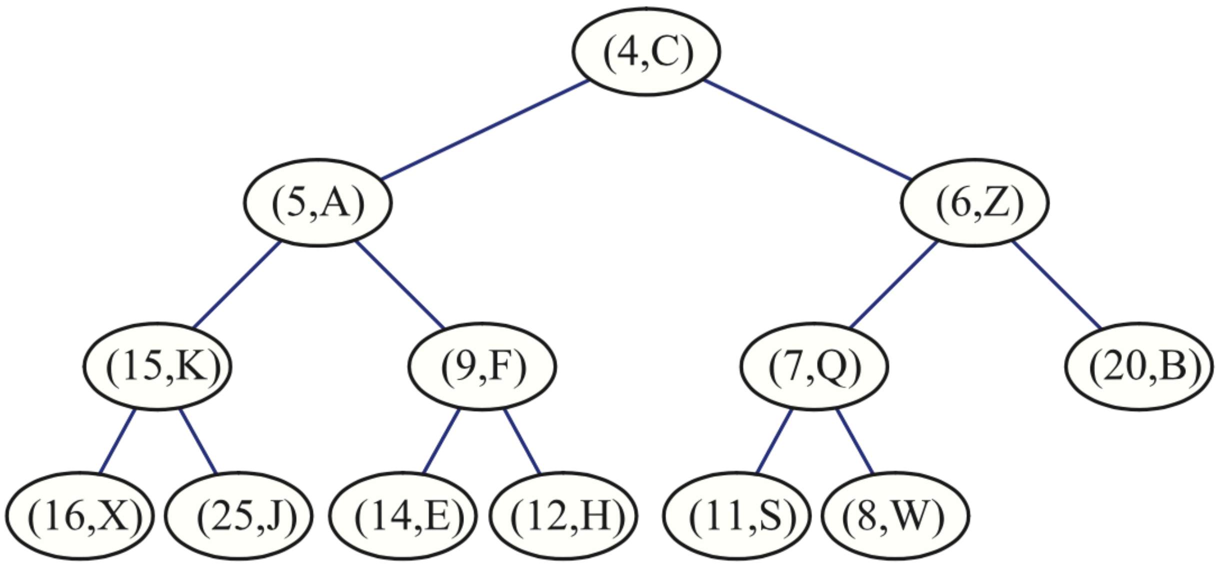
1. Heap-order property

- 1. Heap-Order: for every internal node v other than the root, ***key***(***v***) ≥ ***key***(***parent***(***v***))
  - The keys encountered on a path from the root to a leaf T are nondecreasing
  - A minimum key: always at the root



2. Complete binary tree property

- Complete Binary Tree
  - Roughly speaking, every level, except for the last level, is completely filled, and all nodes in the last level are as far left as possible.
- let h be the height of the heap
  - for $i = 0, \dots, h-1$, there are 2^i nodes of depth i
  - at depth $h-1$, the internal nodes are to the left of the external nodes

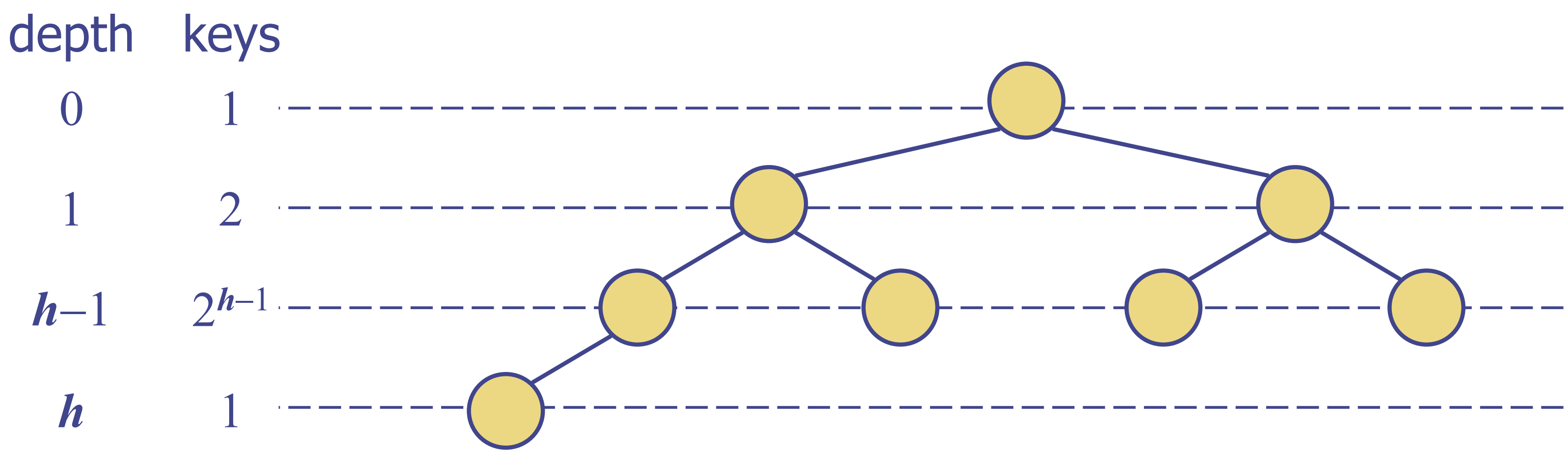


Height of a Heap of *n* elements

- Theorem: A heap storing n keys has height $O(\log n)$

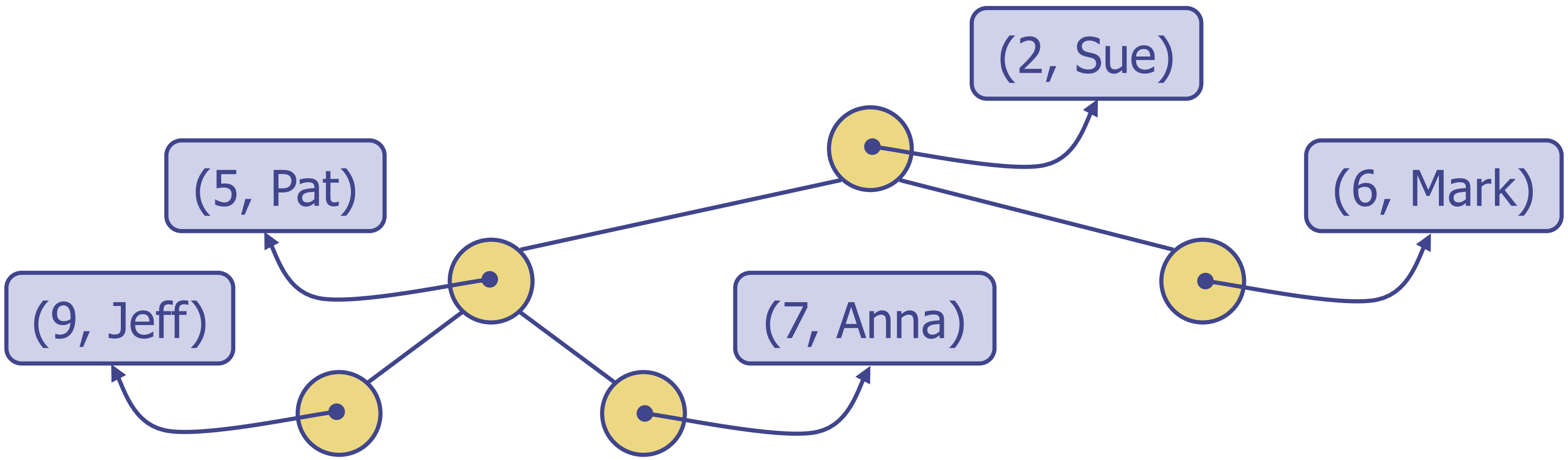
  Proof: (we apply the complete binary tree property)

  - Let h be the height of a heap storing n keys
  - Since there are 2^i keys at depth $i = 0, \ldots, h-1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
  - Thus, $n \geq 2^h$, i.e., $h \leq \log n$



Heaps and Priority Queues

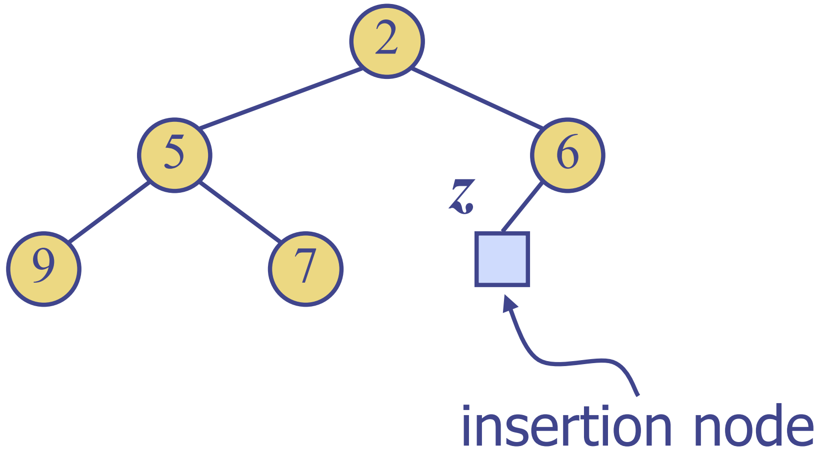
- We can use a heap to implement a priority queue
  - We say "heap-based PQ implementation"
- We store a (key, element) item at each internal node
- We keep track of the position of the last node
  - I am able to know who is the last node in O(1) time
  - Easy

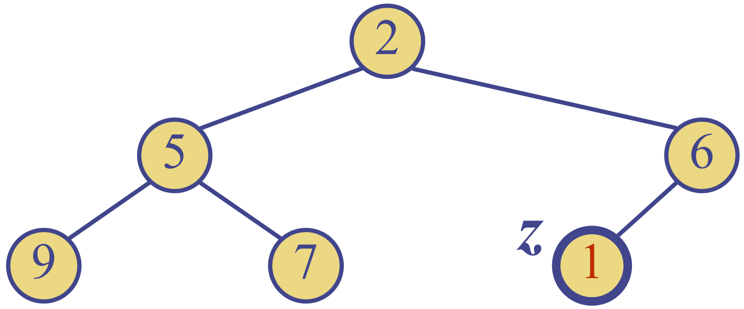


Insertion into a Heap

- Method **insert** of the priority queue ADT corresponds to the insertion of a key k to the heap

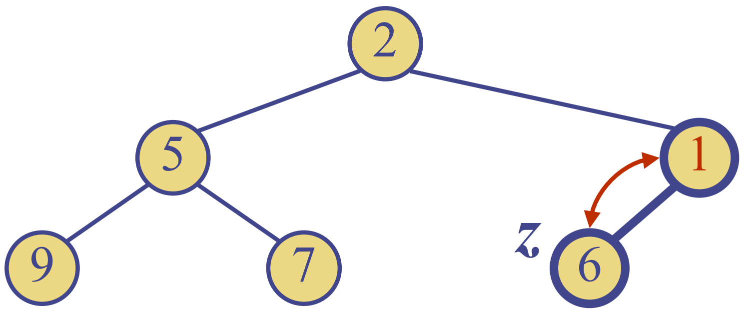
- The insertion algorithm consists of three steps
  - Find the insertion node z (the new last node)
    - How? discussed later
  - Store k at z
  - Restore the heap-order property (discussed next)

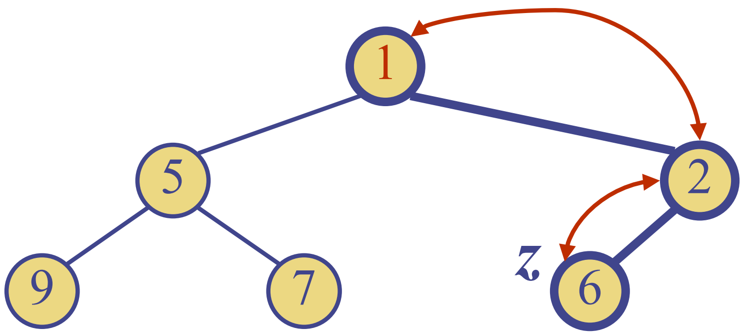




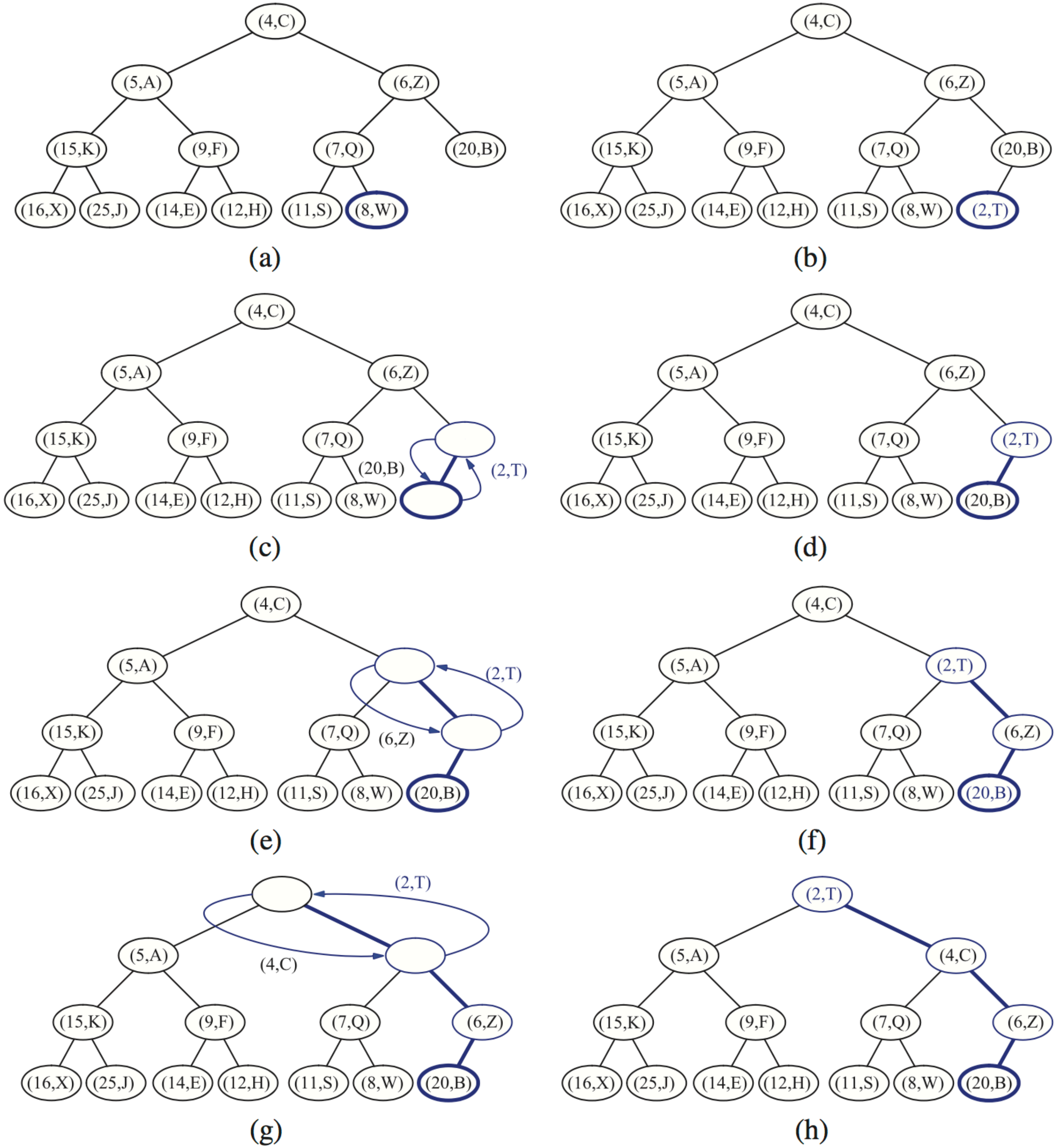
Upheap

- After the insertion of a new key k, the heap-order property may be violated
- Algorithm upheap restores the heap-order property by swapping k along an upward path from the insertion node
- Upheap terminates when the key k reaches the root or a node whose parent has a key smaller than or equal to k
- Since a heap has height $O(\log n)$, upheap runs in $O(\log n)$ time



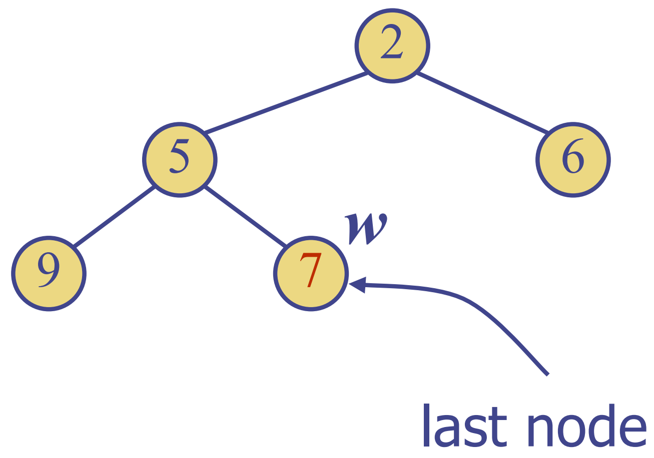


Insert: (2,T)

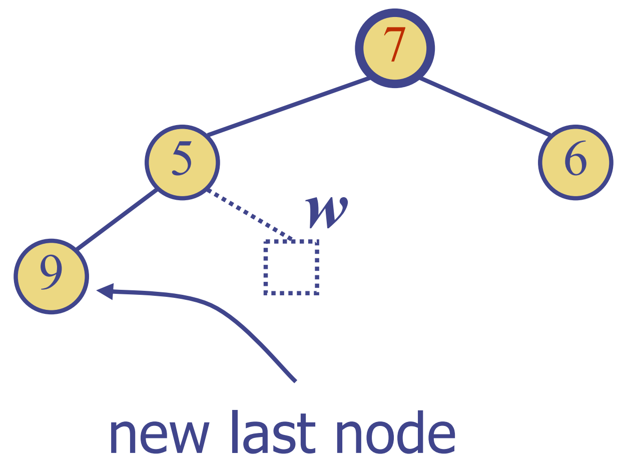

(4,C) (5,A) (6,Z) (15,K) (9,F) (7,Q) (20,B) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W)
(a)
(4,C) (5,A) (6,Z) (15,K) (9,F) (7,Q) (20,B) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (2,T)
(b)
(4,C) (5,A) (6,Z) (15,K) (9,F) (7,Q) (20,B) (2,T) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W)
(c)
(4,C) (5,A) (6,Z) (15,K) (9,F) (7,Q) (2,T) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (20,B)
(d)
(4,C) (5,A) (2,T) (15,K) (9,F) (7,Q) (6,Z) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (20,B)
(e)
(4,C) (5,A) (2,T) (15,K) (9,F) (7,Q) (6,Z) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (20,B)
(f)
(2,T) (5,A) (4,C) (15,K) (9,F) (7,Q) (6,Z) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (20,B)
(g)
(2,T) (5,A) (4,C) (15,K) (9,F) (7,Q) (6,Z) (16,X) (25,J) (14,E) (12,H) (11,S) (8,W) (20,B)
(h)

Removal from a Heap

- Method **removeMin** of the priority queue ADT corresponds to the removal of the root key from the heap

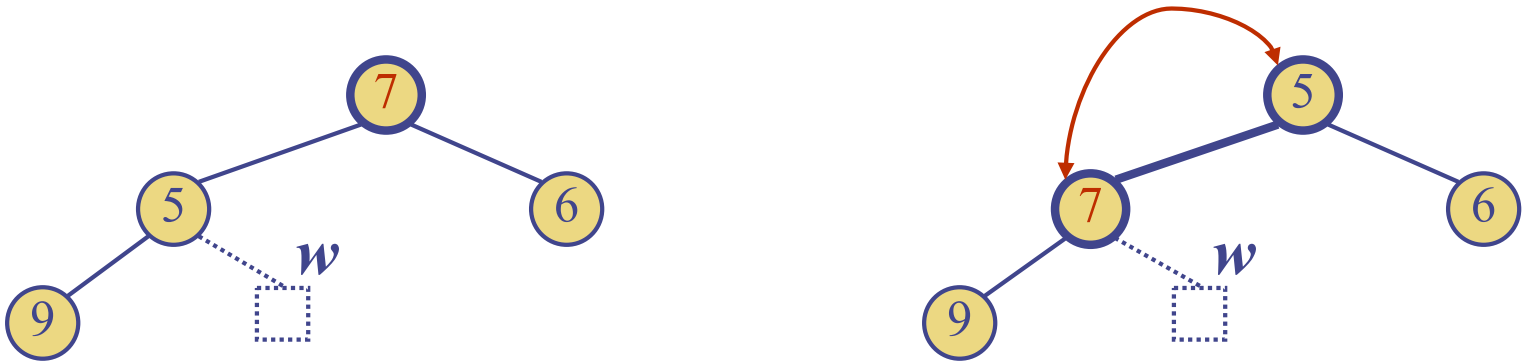


- The removal algorithm consists of three steps
  - Replace the root key with the key of the last node *w*
  - Remove *w*
  - Restore the heap-order property (discussed next)

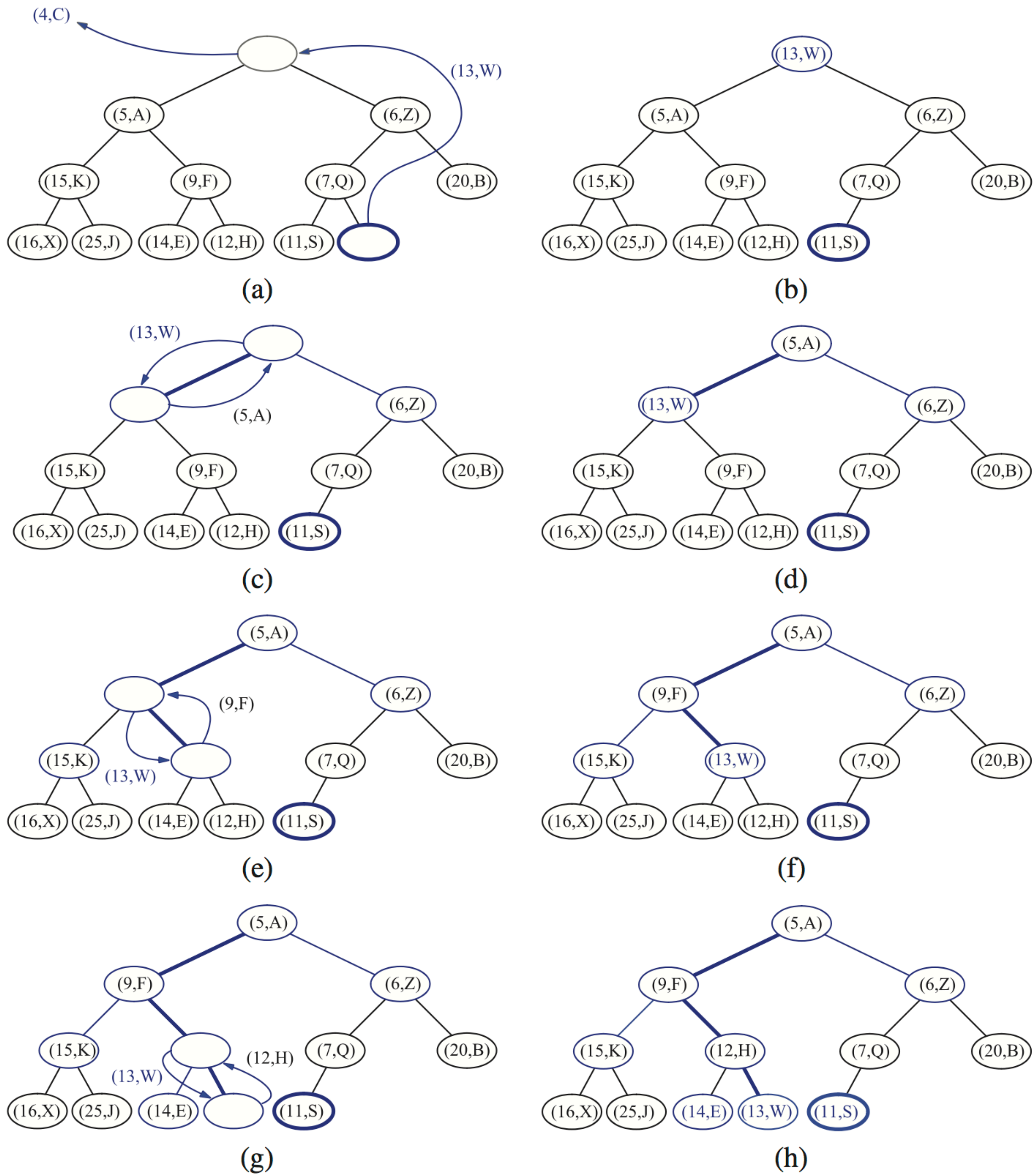


Downheap

- After replacing the root key with the key k of the last node, the heap-order property may be violated
- Algorithm downheap restores the heap-order property by swapping key k along a downward path from the root (but which path?)
- Upheap terminates when key k reaches a leaf or a node whose children have keys greater than or equal to k
- Since a heap has height $O(\log n)$, downheap runs in $O(\log n)$ time



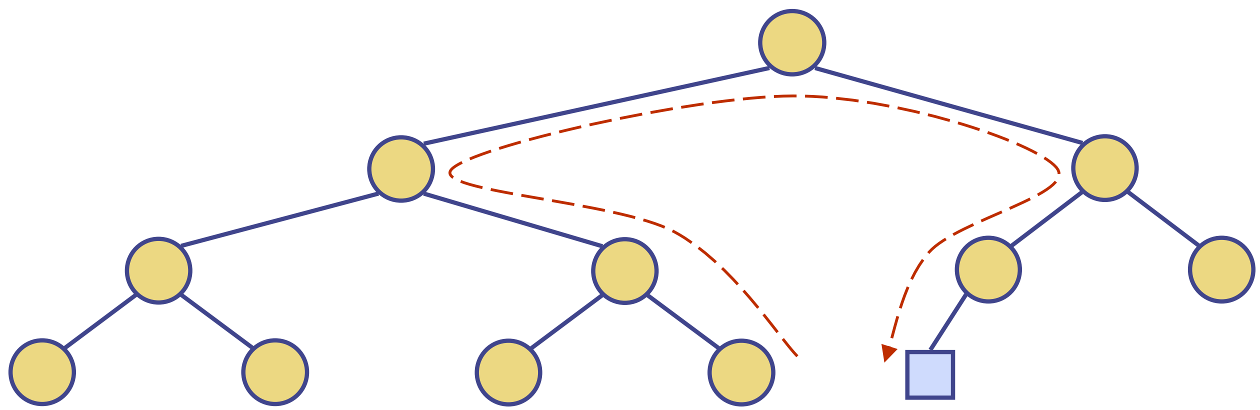
removeMin



Updating the Last Node

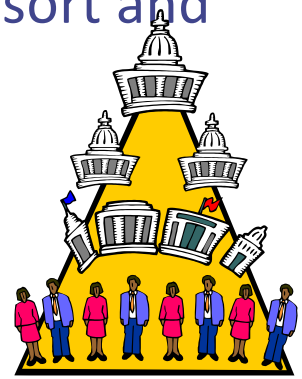
- How can we find the insertion node (a new last node)?
  - The insertion node can be found by traversing a path of $O(\log n)$ nodes
  - (1) Go up until a left child or the root is reached
  - (2) If a left child is reached, go to the right child
  - (3) Go down left until a leaf is reached

- Similar algorithm for updating the last node after a removal



Heap-Sort

- Consider a priority queue with n items implemented by means of a heap
  - the space used is $O(n)$
  - methods insert and removeMin take $O(\log n)$ time
  - methods size, empty, and min take time $O(1)$ time
- Using a heap-based priority queue, we can sort a sequence of n elements in $O(n \log n)$ time
  - Construction: n insertions
  - Actual sorting: n removals
- The resulting algorithm is called heap-sort
- Heap-sort is much faster than quadratic sorting algorithms, such as insertion-sort and selection-sort



Sequence-based vs. Heap-based

Sequence-based

Operation	*Unsorted List*	*Sorted List*
size, empty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

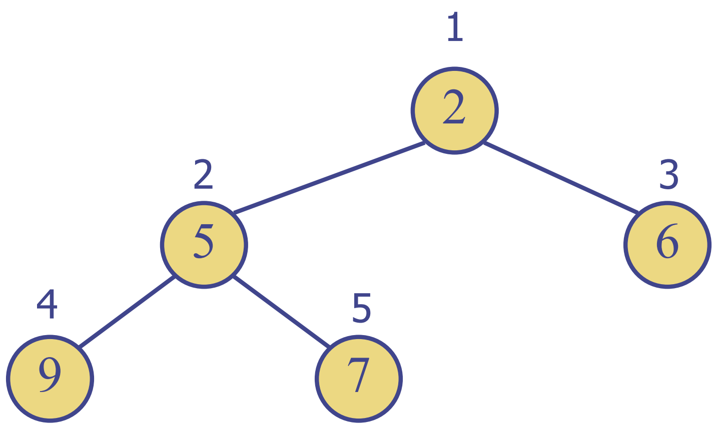
Heap-based

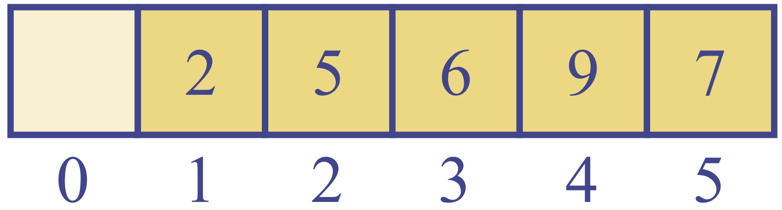
Operation	*Time*
size, empty	$O(1)$
min	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$

How do we remove "stupid repetition"?

Vector-based Heap Implementation

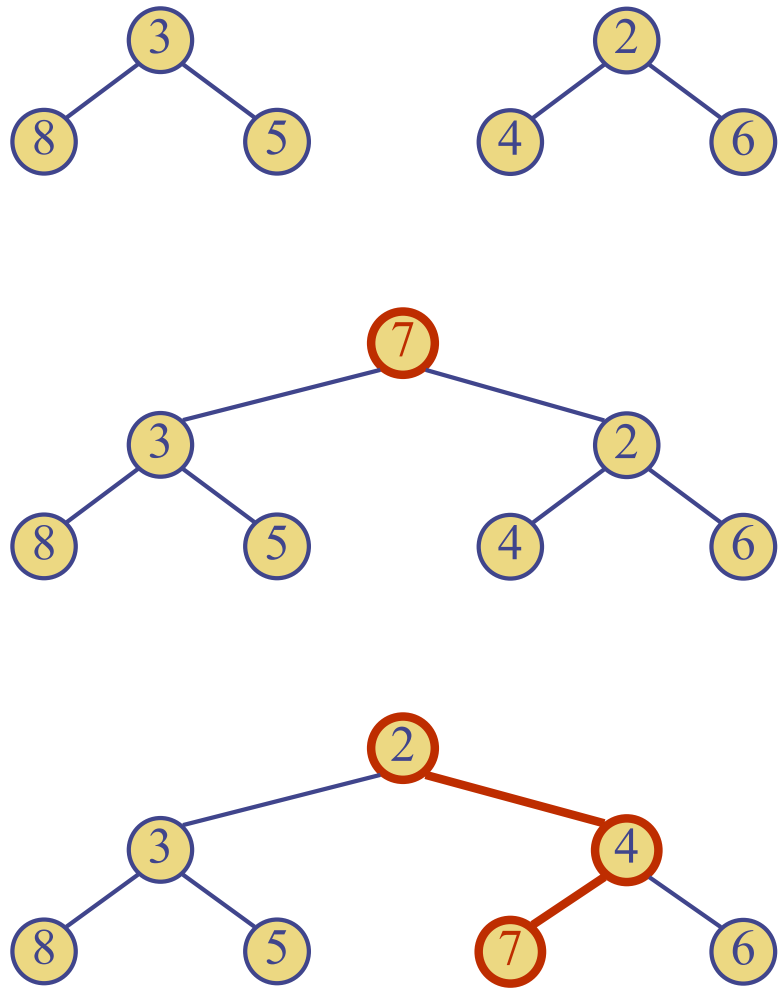
- We can represent a heap with n keys by means of a vector of length $n + 1$
- For the node at rank i
  - the left child is at rank $2i$
  - the right child is at rank $2i + 1$
- Links between nodes are not explicitly stored
- The cell of at rank 0 is not used
- Operation insert corresponds to inserting at rank $n + 1$
- Operation removeMin corresponds to removing at rank n





Merging Two Heaps

- We are given two heaps and a key ***k***
- We create a new heap with the root node storing ***k*** and with the two heaps as subtrees
- We perform downheap to restore the heap-order property



Questions?