Minimum Spanning Trees

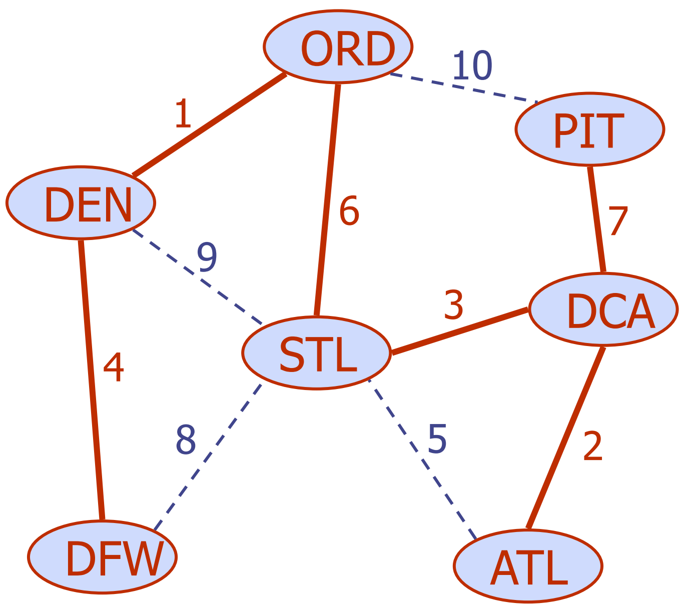
Minimum Spanning Trees

Spanning subgraph
- Subgraph of a graph G containing all the vertices of G

Spanning tree
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)
- Spanning tree of a weighted graph with minimum total edge weight

◆ Applications
- Communications networks
- Transportation networks



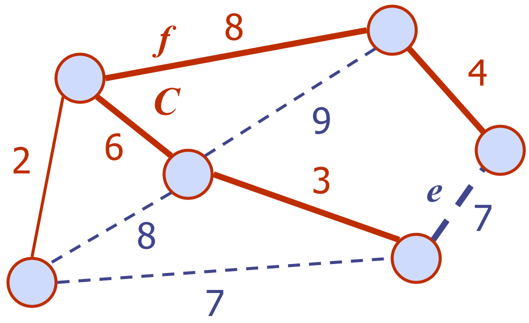
Cycle Property

Cycle Property:

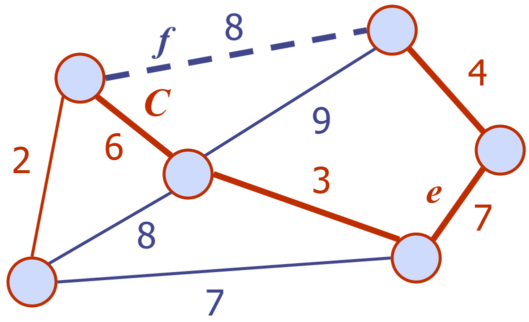
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and C let be the cycle formed by e with T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

- By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f



Replacing f with e yields a better spanning tree



Partition Property

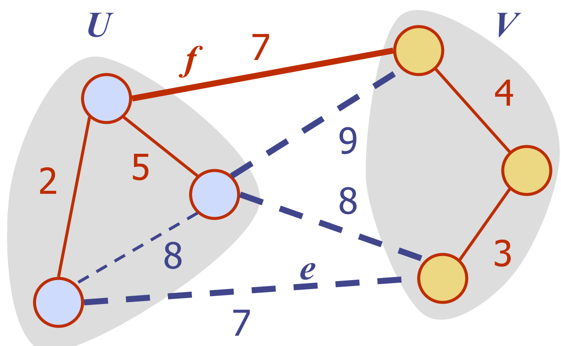
Partition Property:

- Consider a partition of the vertices of G into subsets U and V
- Let e be an edge of minimum weight across the partition
- There is a minimum spanning tree of G containing edge e

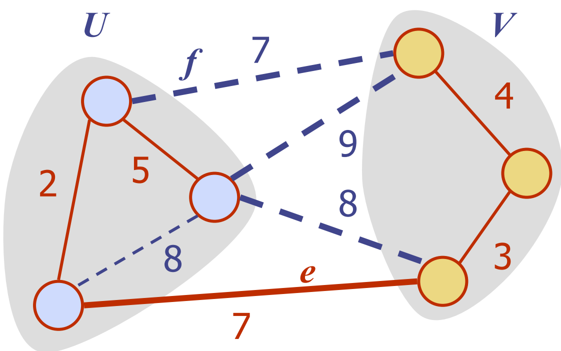
Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property,

  $weight(f) \leq weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e

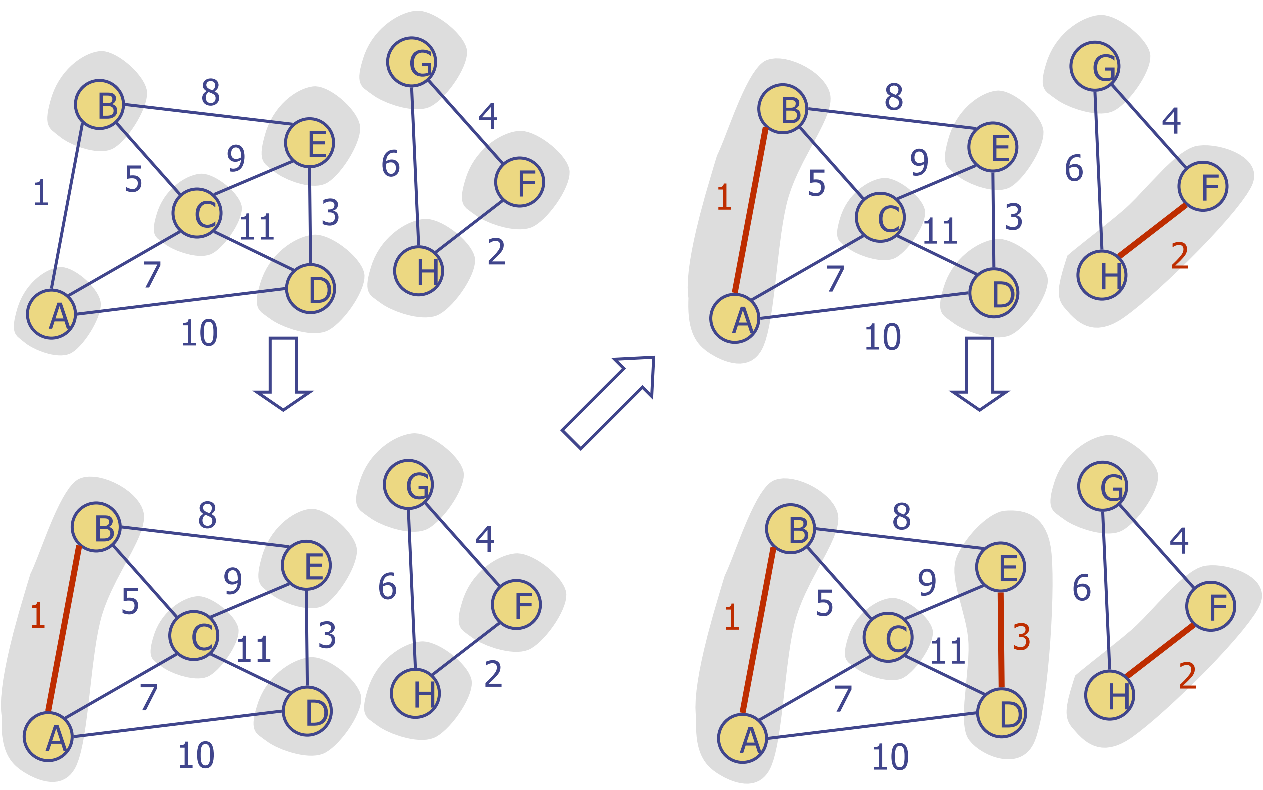


Replacing f with e yields another MST

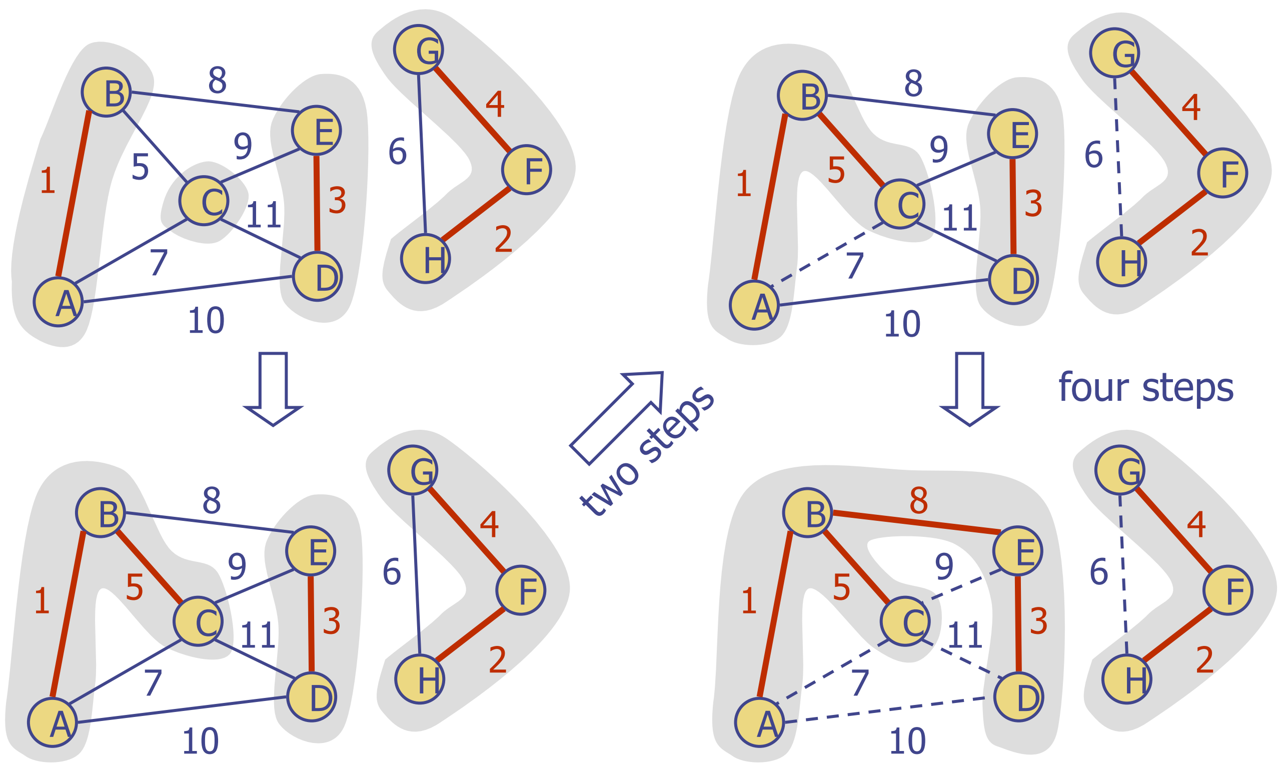


Kruskal's Algorithm

Kruskal's Algorithm: Example



Example (contd.)



Kruskal’s Algorithm

- Maintain a partition of the vertices into clusters
  - Initially, single-vertex clusters
  - Keep an MST for each cluster
  - Merge “closest” clusters and their MSTs
- A priority queue stores the edges outside clusters
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - One cluster and one MST (if connected)

Algorithm *KruskalMST*(*G*)
  for each vertex *v* in *G* **do**
    Create a cluster consisting of *v*
  let *Q* be a priority queue.
  Insert all edges into *Q*
  $T \leftarrow \emptyset$
  {*T* is the union of the MSTs of the clusters}
  while *T* has fewer than $n - 1$ edges **do**
    $e \leftarrow$ *Q.removeMin*().*getValue*()
    $[u, v] \leftarrow$ *G.endVertices*(*e*)
    $A \leftarrow$ *getCluster*(*u*)
    $B \leftarrow$ *getCluster*(*v*)
    if $A \neq B$ **then**
      Add edge *e* to *T*
      mergeClusters(*A*, *B*)
  return *T*

Data Structure for Kruskal's Algorithm

- The algorithm maintains a forest of trees
- A priority queue extracts the edges by increasing weight
- An edge is accepted it if connects distinct trees

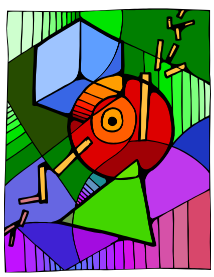
- We need a data structure that maintains a partition, i.e., a collection of disjoint sets
  - To do this, we need a data structure for a set
  - These are covered in Ch. 11.4 (Page 533)

Set Operations

- We represent a set by the sorted sequence of its elements

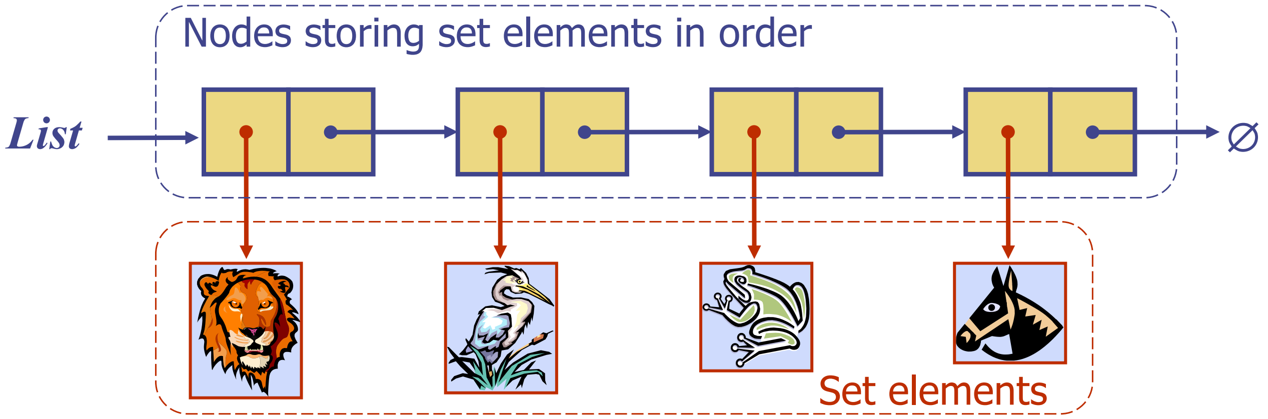
- The basic set operations:
  - union
  - intersection
  - subtraction

- We consider
  - Sequence-based implementation



Example: Storing a Set in a Sorted List

- We can implement a set with a list
- Elements are stored sorted according to some canonical ordering
- The space used is $O(n)$

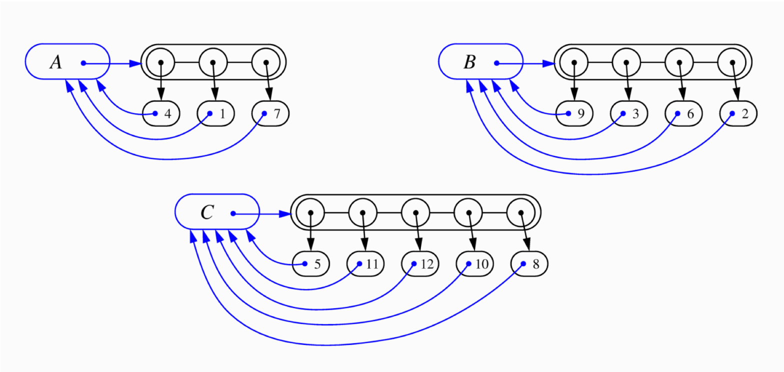


Partitions with Union-Find Operations

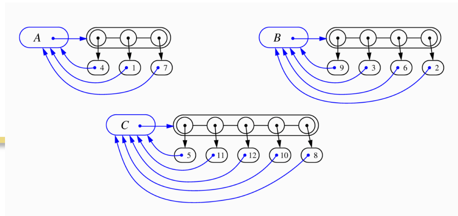
- Partition: A collection of disjoint sets
- Partition ADT needs to support the following functions:
  - makeSet(x): Create a singleton set containing the element x and return the position storing x in this set
  - union(A,B): Return the set A U B, destroying the old A and B
  - find(p): Return the set containing the element at position p

List-based Partition (1)

- Each set is stored in a sequence (e.g., list)
- Partition: A collection of sequences
- Each element has a reference back to the set
  - Operation find(u): takes O(1) time, and returns the set of which u is a member.
  - Operation union(A,B): we move the elements of the smaller set to the sequence of the larger set and update their references
    - Time for operation union(A,B) is min(|A|, |B|)
    - Worst-case: O(n) for one union operation



List-based Partition (2)



- What about "amortized analysis"? (Page 539)

Proposition 11.9: *Performing a series of n* makeSet, union, *and* find *operations, using the sequence-based implementation above, starting from an initially empty partition takes $O(n \log n)$ time.*

- Clearly, makeSet and find operation → O(n)
- Union operation
  - Each time we move a position from one set to another, the size of the new set at least doubles
  - Thus, each position is moved from one set to another at most *log n* times
  - We assume that the partition is initially empty, there are *O(n)* different elements referenced in the given series of operations. → The total time for all the union operations is *O(n log n)*

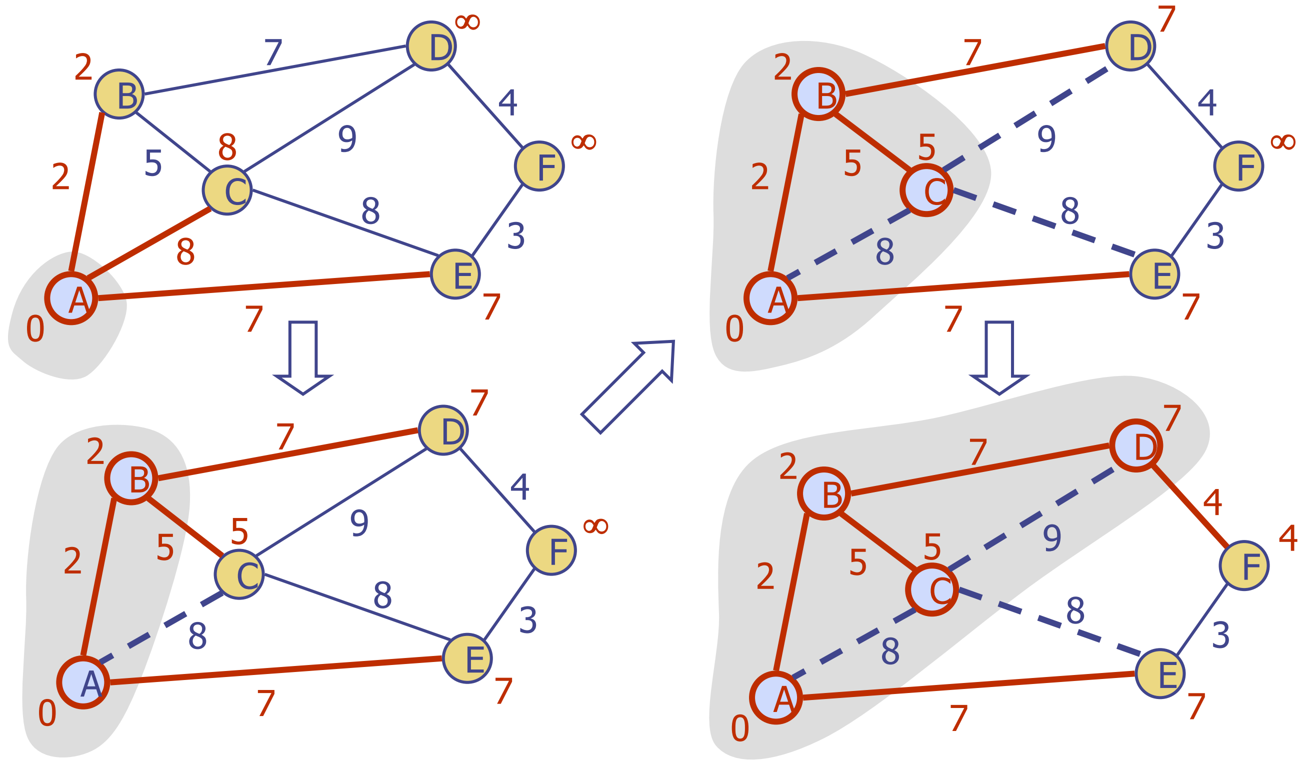
Partition-Based Implementation

- Partition-based version of Kruskal's Algorithm
  - Cluster merges as unions
  - Cluster locations as finds
- Running time $O((n + m) \log n)$
  - PQ operations $O(m \log n)$
    - PQ initialization: O(mlog m)
    - For each while loop
      - O(log m) = O(log n)
  - UF operations $O(n \log n)$

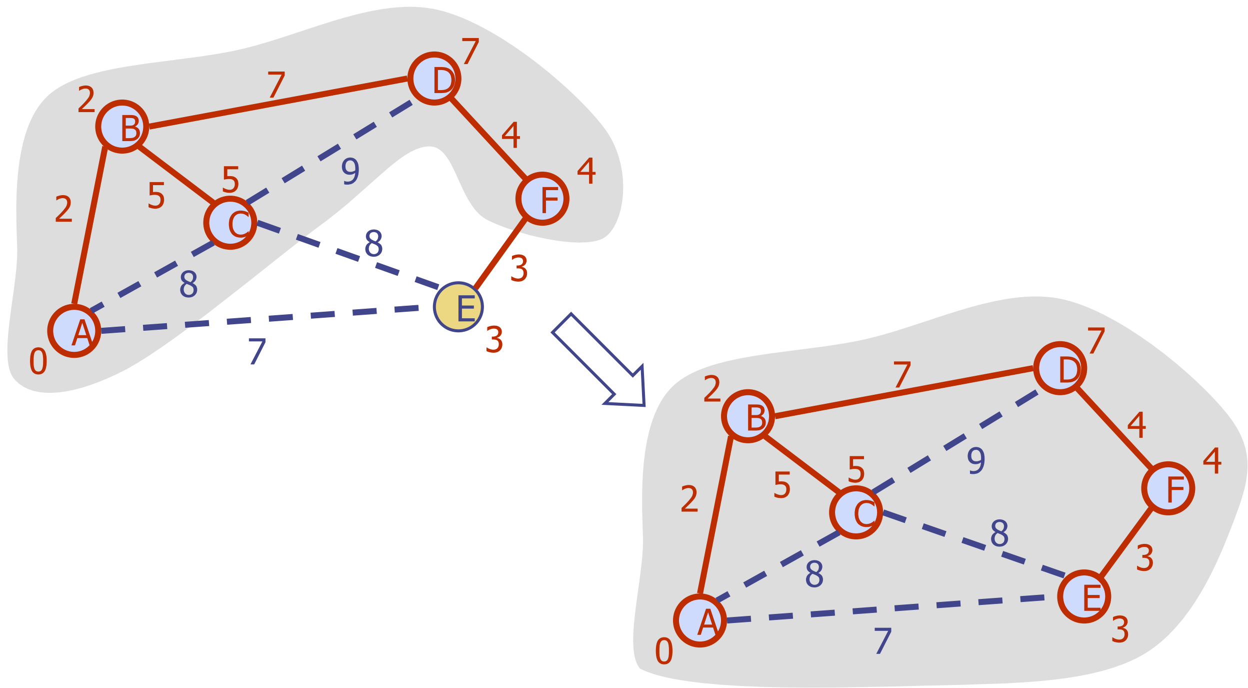
Algorithm *KruskalMST*(*G*)
  Initialize a partition *P*
  for each vertex *v* in *G* **do**
    P.makeSet(*v*)
  let *Q* be a priority queue.
  Insert all edges into *Q*
  $T \leftarrow \varnothing$
  {*T* is the union of the MSTs of the clusters}
  while *T* has fewer than $n - 1$ edges **do**
    $e \leftarrow$ *Q.removeMin*().*getValue*()
    $[u, v] \leftarrow$ *G.endVertices*(*e*)
    $A \leftarrow$ *P.find*(*u*)
    $B \leftarrow$ *P.find*(*v*)
    if $A \neq B$ **then**
      Add edge *e* to *T*
      P.union(*A*, *B*)
  return *T*

Prim-Janik's Algorithm

Example



Example (contd.)



Prim-Jarnik's Algorithm

- Similar to Dijkstra's algorithm
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v label $d(v)$ representing the smallest weight of an edge connecting v to a vertex in the cloud

(see the difference from Dijkstra's algorithm?)

- At each step:
  - We add to the cloud the vertex u outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u

Prim-Jarnik's Algorithm (cont.)

- A heap-based adaptable priority queue with location-aware entries stores the vertices outside the cloud
  - Key: distance
  - Value: vertex
  - Recall that method *replaceKey*(l,k) changes the key of entry l
- We store three labels with each vertex:
  - Distance
  - Parent edge in MST
  - Entry in priority queue

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      v.setDistance(0)
    else
      v.setDistance(∞)
    v.setParent(∅)
    l ← Q.insert(v.getDistance(), v)
    v.setLocator(l)
  while ¬Q.empty()
    l ← Q.removeMin()
    u ← l.getValue()
    for all e ∈ u.incidentEdges()
      z ← e.opposite(u)
      r ← e.weight()
      if r < z.getDistance()
        z.setDistance(r)
        z.setParent(e)
        Q.replaceKey(z.getEntry(), r)
```

Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time (total $n\ O(\log n)$)
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

Questions?