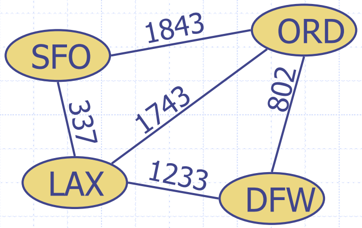
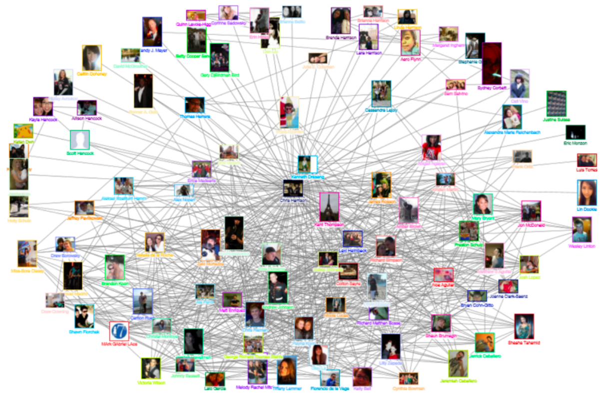
# Graphs: Basics






# Real Life Examples

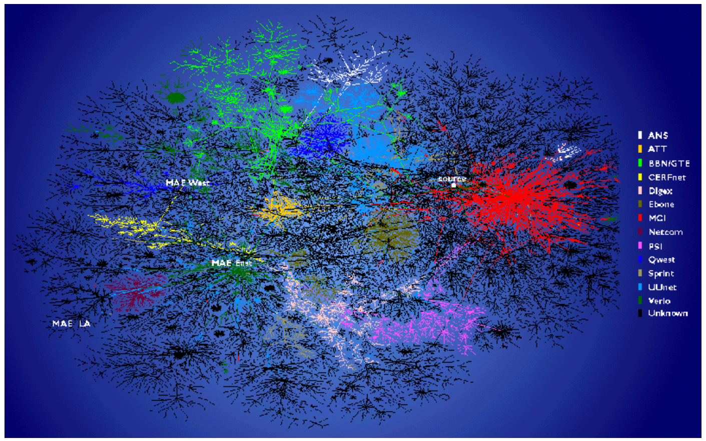
- On-line/Off-line Social Network





# Real Life Examples

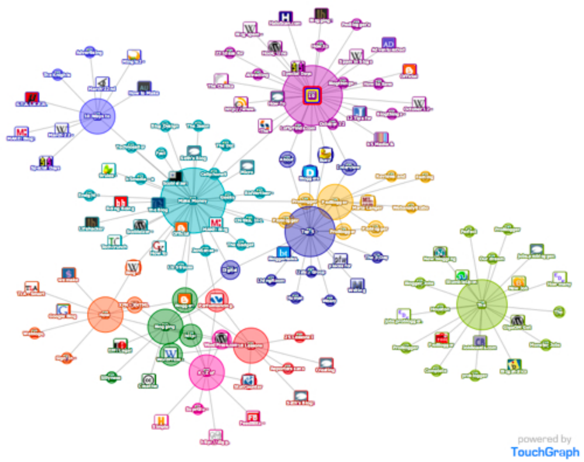
- Internet Connectivity





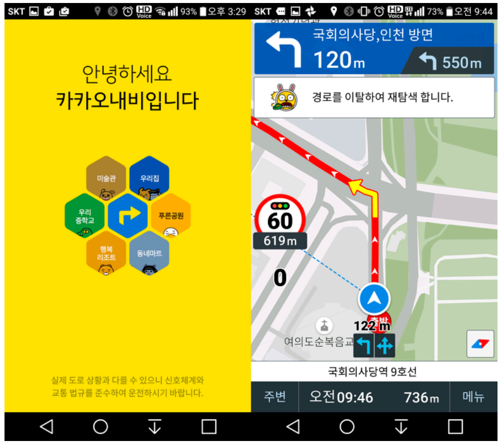

# Real Life Examples

- WebBlog Connections



# Real Life Examples

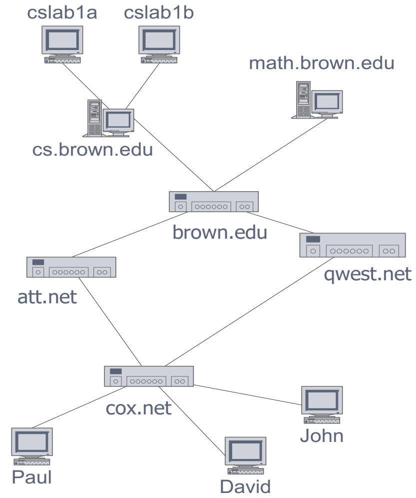
- Navigator






# Other Applications

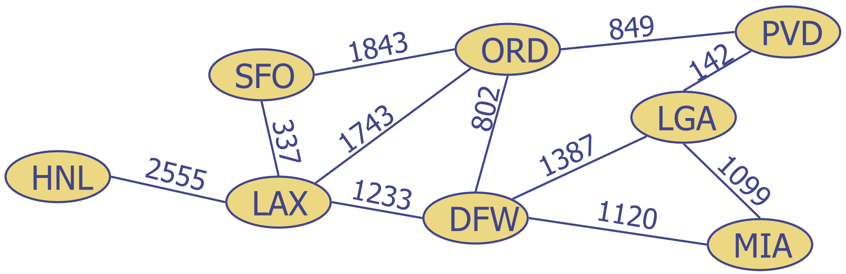
- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram






# Graphs

- A graph is a pair $(V, E)$, where
  - $V$ is a set of nodes, called vertices
  - $E$ is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



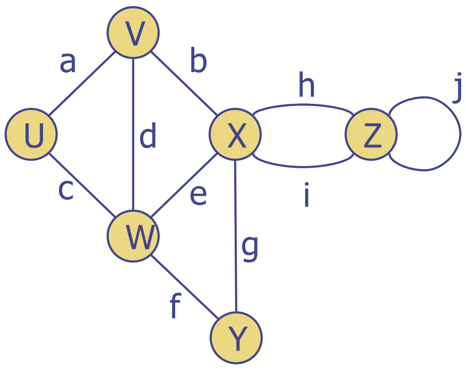



# Edge Types

- Directed edge
  - ordered pair of vertices $(u,v)$
  - first vertex $u$ is the origin
  - second vertex $v$ is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices $(u,v)$
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network


# Terminology

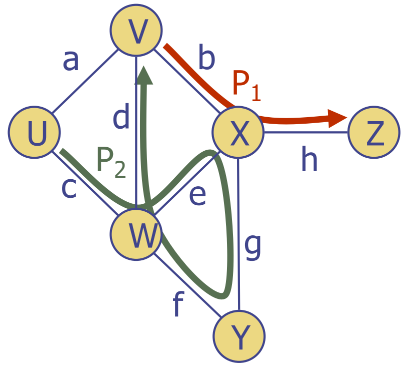
- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop






# Terminology (cont.)

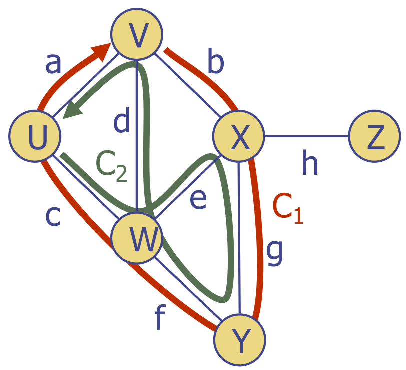
- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - $P_1$=(V,b,X,h,Z) is a simple path
  - $P_2$=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple






# Terminology (cont.)

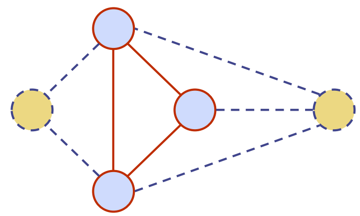
- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - $C_1$=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - $C_2$=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple
- Note) Tree is a graph without cycles



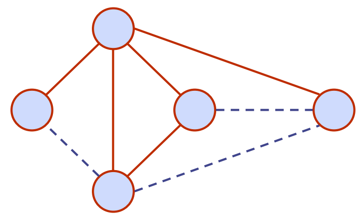



# Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



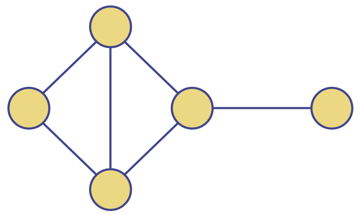
Subgraph



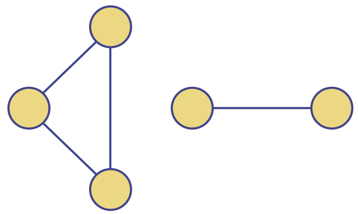
Spanning subgraph

# Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G
- "Maximal"?



Connected graph



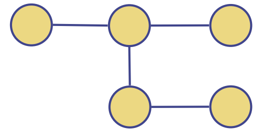
Non connected graph with two connected components



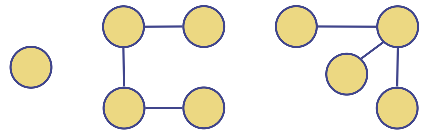
# Trees and Forests

- A (free) tree is an undirected graph T such that
  - T is connected
  - T has no cycles

  This definition of tree is different from the one of a rooted tree
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



Tree

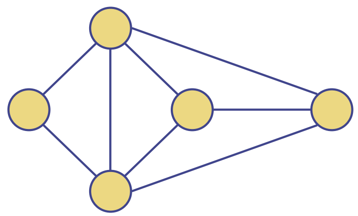


Forest

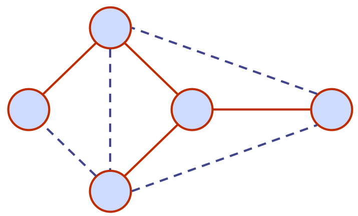


# Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



Graph



Spanning tree



# Some Properties for Undirected Graphs

Property 1

$\sum_v \deg(v) = 2m$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

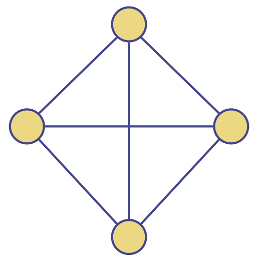
$m \leq n\,(n-1)/2$

Proof: each vertex has degree at most $(n-1)$

What is the bound for a directed graph?

Notation

| | |
|---|---|
| $n$ | number of vertices |
| $m$ | number of edges |
| $\deg(v)$ | degree of vertex $v$ |



Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

# Main Methods of the Graph ADT

- Vertices and edges
  - are positions
  - store elements
- Accessor methods
  - e.endVertices(): a list of the two endvertices of e
  - e.opposite(v): the vertex opposite of v on e
  - u.isAdjacentTo(v): true iff u and v are adjacent
  - *v: reference to element associated with vertex v
  - *e: reference to element associated with edge e
- Update methods
  - insertVertex(o): insert a vertex storing element o
  - insertEdge(v, w, o): insert an edge (v,w) storing element o
  - eraseVertex(v): remove vertex v (and its incident edges)
  - eraseEdge(e): remove edge e
- Iterable collection methods
  - incidentEdges(v): list of edges incident to v
  - vertices(): list of all vertices in the graph
  - edges(): list of all edges in the graph



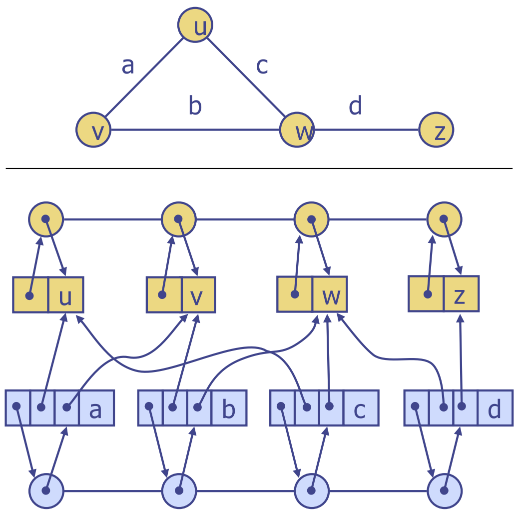
What is a data structure to represent a graph?

We will discuss three ways



# 1. Edge List Structure

- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence (e.g., list)
  - sequence of vertex objects
- Edge sequence (e.g., list)
  - sequence of edge objects





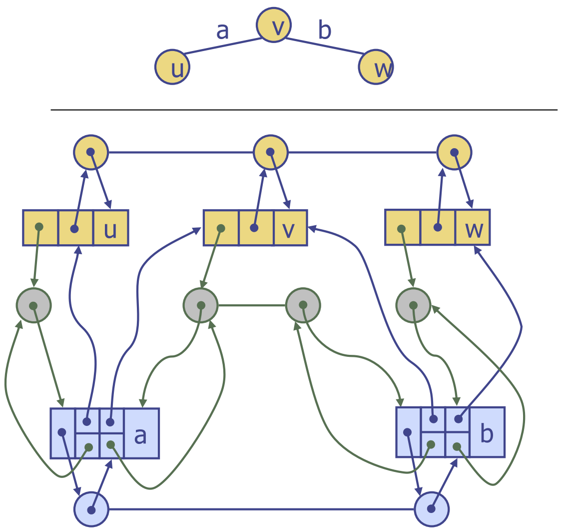

# Performance

| ▪ *n* vertices, *m* edges<br>▪ no parallel edges<br>▪ no self-loops | Edge List | Adjacency List | Adjacency Matrix |
|---|---|---|---|
| Space | $n + m$ | $n + m$ | $n^2$ |
| *v*.incidentEdges() | $m$ | deg(*v*) | $n$ |
| *u*.isAdjacentTo (*v*) | $m$ | min(deg(*v*), deg(*w*)) | 1 |
| insertVertex(*o*) | 1 | 1 | $n^2$ |
| insertEdge(*v, w, o*) | 1 | 1 | 1 |
| eraseVertex(*v*) | $m$ | deg(*v*) | $n^2$ |
| eraseEdge(*e*) | 1 | 1 | 1 |

- v.incidentEdges() and u.isAdjacneTo(v)
  - Need to check all the edges

# 2. Adjacency List Structure

- Basic: Edge list structure
- Supports direct access to the incident edges from a node
  - Incidence edge sequence for each vertex
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices
- Provides direct access
  - From the edges to the vertices
  - From the vertices to their incident edges






# Performance

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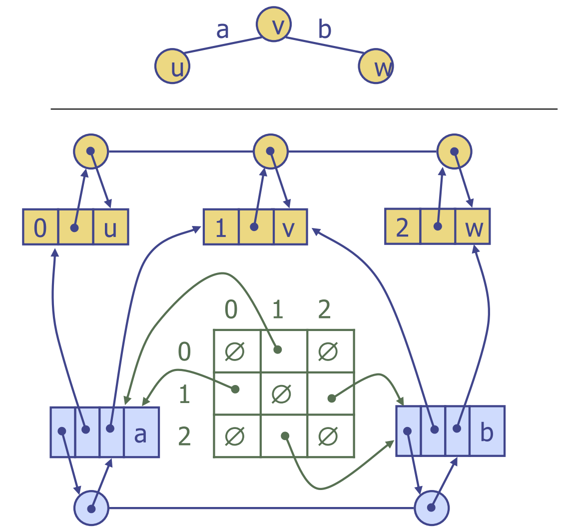
|---|---|---|---|
| Space | $n + m$ | $n + m$ | $n^2$ |
| $v$.incidentEdges() | $m$ | deg($v$) | $n$ |
| $u$.isAdjacentTo ($v$) | $m$ | min(deg($v$), deg($w$)) | 1 |
| insertVertex($o$) | 1 | 1 | $n^2$ |
| insertEdge($v, w, o$) | 1 | 1 | 1 |
| eraseVertex($v$) | $m$ | deg($v$) | $n^2$ |
| eraseEdge($e$) | 1 | 1 | 1 |

- v.incidentEdges(): direct access to incident edges
- u.isAdjacentTo(v):



# 3. Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non nonadjacent vertices
- The "old fashioned" version just has 0 for no edge and 1 for edge






# Performance

| ▪ $n$ vertices, $m$ edges<br>▪ no parallel edges<br>▪ no self-loops | Edge List | Adjacency List | Adjacency Matrix |
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| insertEdge($v, w, o$) | 1 | 1 | 1 |
| eraseVertex($v$) | $m$ | deg($v$) | $n^2$ |
| eraseEdge($e$) | 1 | 1 | 1 |

- v.incidentEdges(): matrix row check
- u.isAdjacentTo(v): using v's key

# Performance

| ▪ $n$ vertices, $m$ edges<br>▪ no parallel edges<br>▪ no self-loops | Edge List | Adjacency List | Adjacency Matrix |
|---|---|---|---|
| Space | $n + m$ | $n + m$ | $n^2$ |
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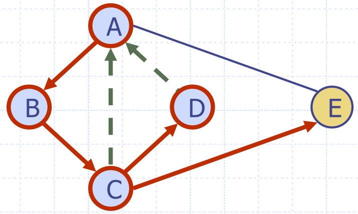
| insertEdge($v, w, o$) | 1 | 1 | 1 |
| eraseVertex($v$) | $m$ | deg($v$) | $n^2$ |
| eraseEdge($e$) | 1 | 1 | 1 |

- v.incidentEdges(): direct access to incident edges
- u.isAdjacentTo(v):



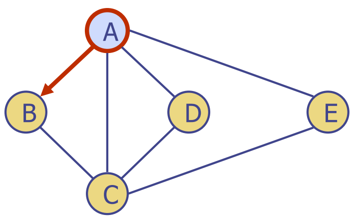
# Depth-First Search





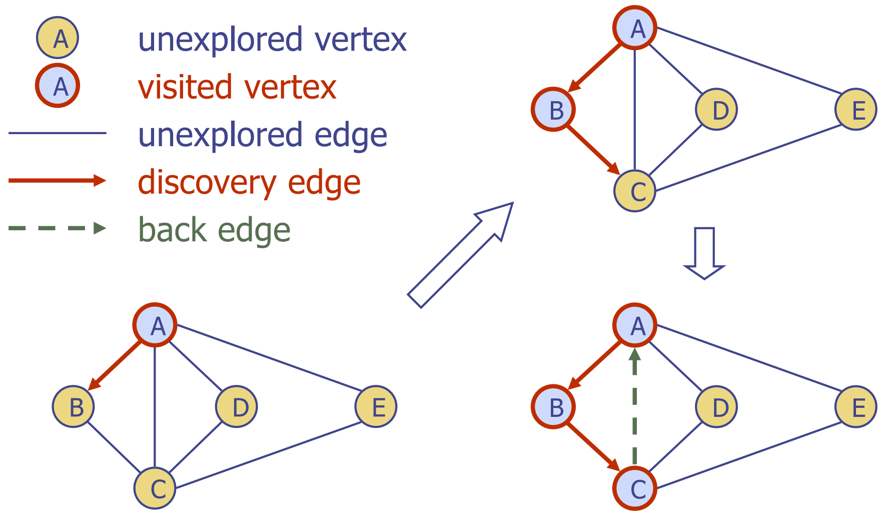

# Depth-First Search

- Depth-first search (DFS) is a general technique for traversing a graph
- Why is this traversal important?
- Let's first see the example

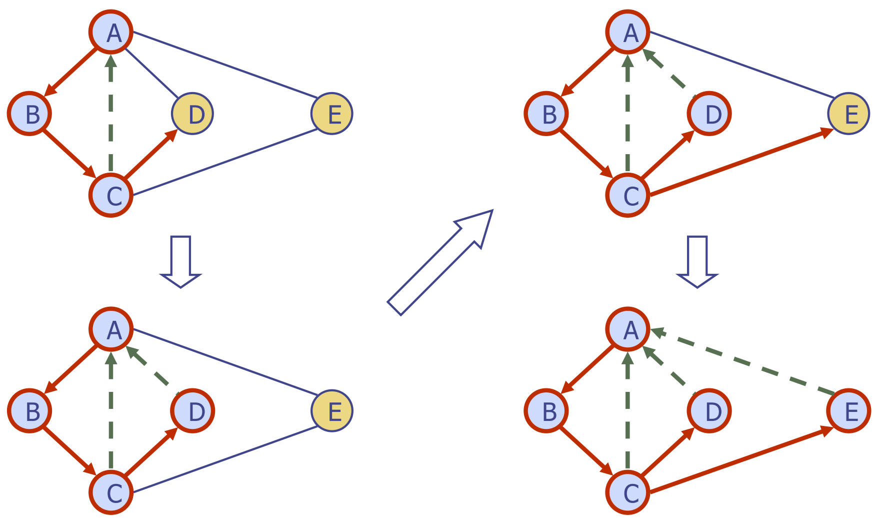





# Example



## Example (cont.)



One implication: discovery edges form a spanning tree.



## Depth-First Search

- A DFS traversal of a graph G
  - Visits all the vertices and edges of G
  - Determines whether G is connected (how?)
  - Computes the connected components of G (how?)
  - Computes a spanning forest of G
- DFS on a graph with $n$ vertices and $m$ edges takes $O(n + m)$ time
- DFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Find a cycle in the graph



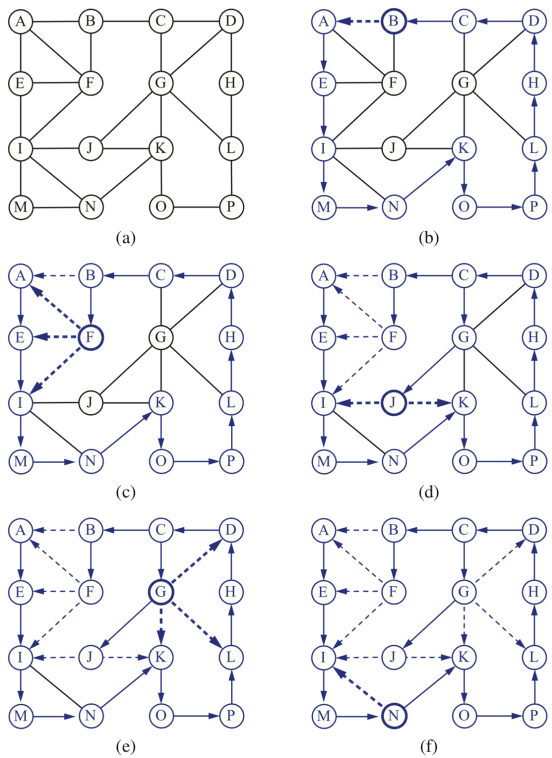
## DFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

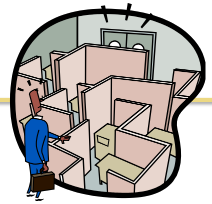
```
Algorithm DFS(G)
    Input graph G
    Output labeling of the edges of G
        as discovery edges and
        back edges
  for all u ∈ G.vertices()
    u.setLabel(UNEXPLORED)
  for all e ∈ G.edges()
    e.setLabel(UNEXPLORED)
  for all v ∈ G.vertices()
    if v.getLabel() = UNEXPLORED
        DFS(G, v)
```

```
Algorithm DFS(G, v)
  Input graph G and a start vertex v of G
  Output labeling of the edges of G
    in the connected component of v
    as discovery edges and back edges
  v.setLabel(VISITED)
  for all e ∈ G.incidentEdges(v)
    if e.getLabel() = UNEXPLORED
      w ← e.opposite(v)
      if w.getLabel() = UNEXPLORED
        e.setLabel(DISCOVERY)
        DFS(G, w)
      else
        e.setLabel(BACK)
```

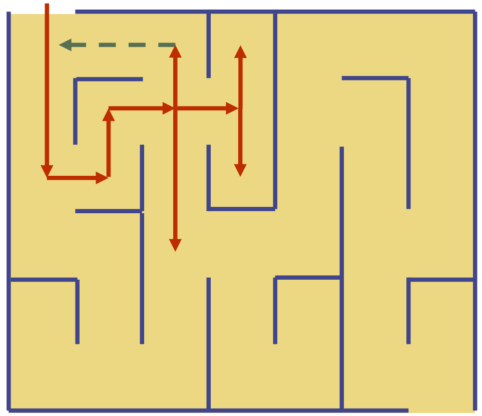




# DFS and Maze Traversal



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)





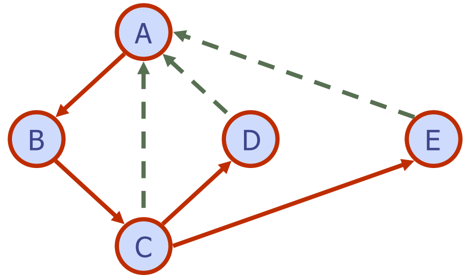
# Properties of DFS

Property 1

$DFS(G, v)$ visits all the vertices and edges in the connected component of $v$

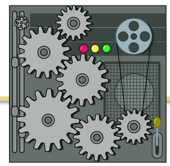
Property 2

The discovery edges labeled by $DFS(G, v)$ form a spanning tree of the connected component of $v$






# Analysis of DFS



- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or BACK
- Method incidentEdges is called once for each vertex
  - Complexity of v.incidentEdges: deg(v)
- DFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$



# Path Finding

- We can specialize the DFS algorithm to find a path between two given vertices $u$ and $z$ using the template method pattern
- We call $DFS(G, u)$ with $u$ as the start vertex
- We use a stack $S$ to keep track of the path between the start vertex and the current vertex
- As soon as destination vertex $z$ is encountered, we return the path as the contents of the stack

```
Algorithm pathDFS(G, v, z)
  v.setLabel(VISITED)
  S.push(v)
  if v = z
    return S.elements()
  for all e ∈ v.incidentEdges()
    if e.getLabel() = UNEXPLORED
      w ← e.opposite(v)
      if w.getLabel() = UNEXPLORED
        e.setLabel(DISCOVERY)
        S.push(e)
        pathDFS(G, w, z)
        S.pop(e)
      else
        e.setLabel(BACK)
  S.pop(v)
```

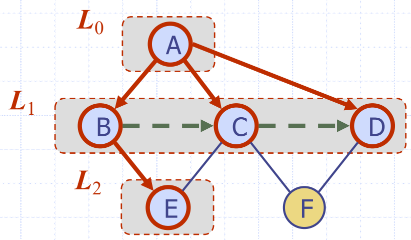
## Cycle Finding

- We can specialize the DFS algorithm to find a simple cycle using the template method pattern
- We use a stack $S$ to keep track of the path between the start vertex and the current vertex
- As soon as a back edge $(v, w)$ is encountered, we return the cycle as the portion of the stack from the top to vertex $w$

```
Algorithm cycleDFS(G, v, z)
  v.setLabel(VISITED)
  S.push(v)
  for all  e ∈ v.incidentEdges()
    if  e.getLabel() = UNEXPLORED
      w ← e.opposite(v)
      S.push(e)
      if w.getLabel() = UNEXPLORED
        e.setLabel(DISCOVERY)
        pathDFS(G, w, z)
        S.pop(e)
      else
        T ← new empty stack
        repeat
          o ← S.pop()
          T.push(o)
        until o = w
        return T.elements()
  S.pop(v)
```



## Breadth-First Search



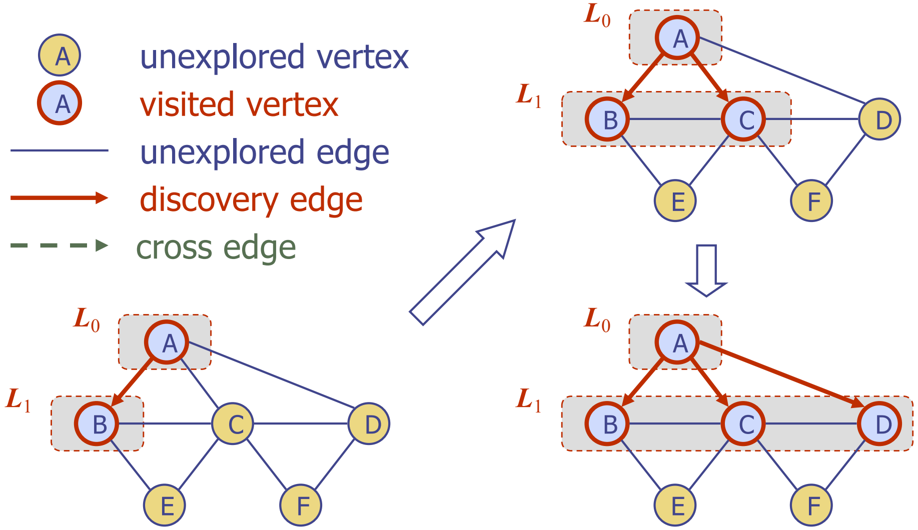



## Breadth-First Search

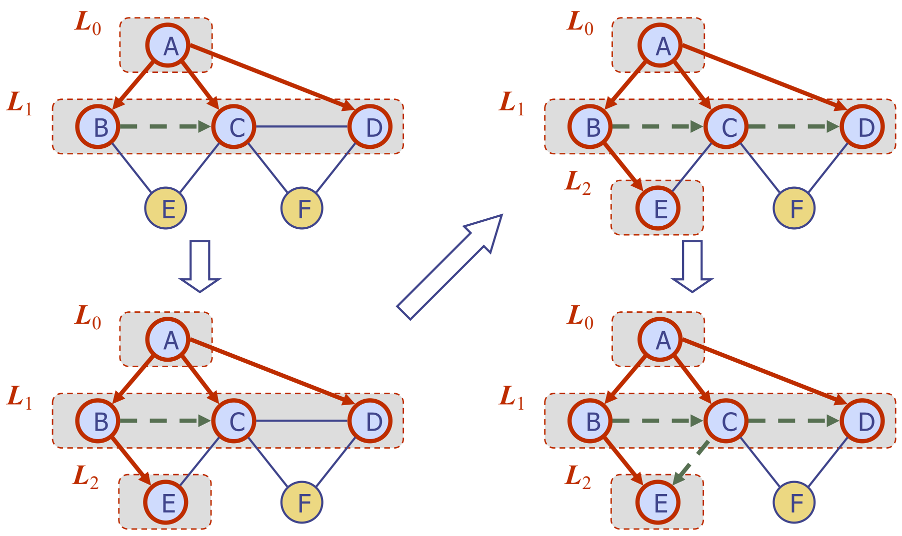
- Breadth-first search (BFS) is another general technique for traversing a graph
- Let's look at the example



## Example

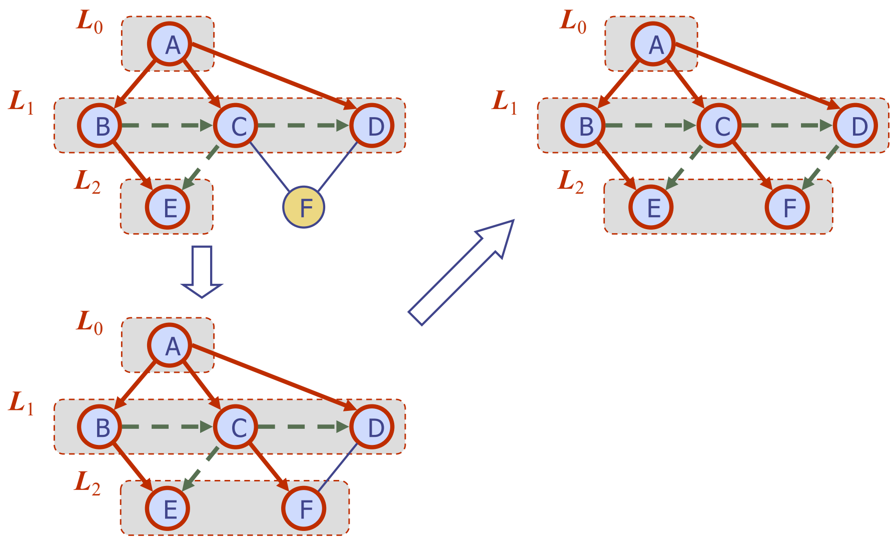


## Example (cont.)






## Example (cont.)






## Breadth-First Search

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  - Computes a spanning forest of G
- BFS on a graph with $n$ vertices and $m$ edges takes $O(n + m)$ time
- BFS can be further extended to solve other graph problems
  - Find and report a path between two given vertices
  - Can label each vertex by the length of a shortest path (in terms of # of edges) from the start vertex s
  - Find a simple cycle, if there is one

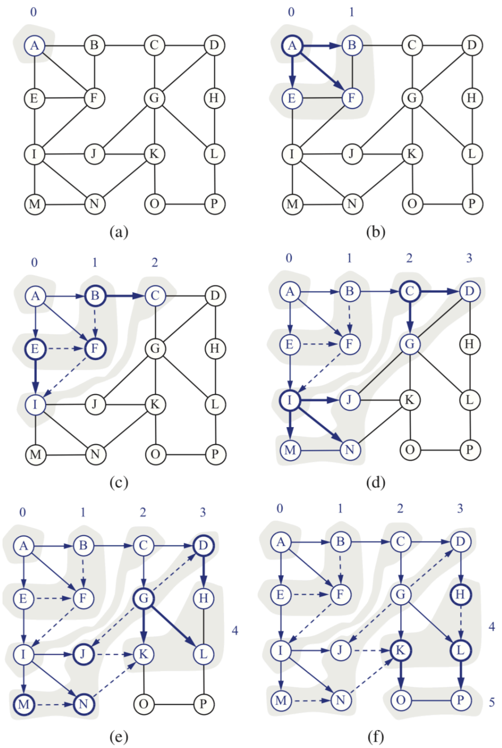


## BFS Algorithm

- The algorithm uses a mechanism for setting and getting "labels" of vertices and edges

```
Algorithm BFS(G)
    Input graph G
    Output labeling of the edges
        and partition of the
        vertices of G
  for all u ∈ G.vertices()
    u.setLabel(UNEXPLORED)
  for all e ∈ G.edges()
    e.setLabel(UNEXPLORED)
  for all v ∈ G.vertices()
    if v.getLabel() = UNEXPLORED
        BFS(G, v)
```

```
Algorithm BFS(G, s)
  L0 ← new empty sequence
  L0.insertBack(s)
  s.setLabel(VISITED)
  i ← 0
  while ¬Li.empty()
    Li+1 ← new empty sequence
    for all v ∈ Li.elements()
      for all e ∈ v.incidentEdges()
        if e.getLabel() = UNEXPLORED
          w ← e.opposite(v)
          if w.getLabel() = UNEXPLORED
            e.setLabel(DISCOVERY)
            w.setLabel(VISITED)
            Li+1.insertBack(w)
          else
            e.setLabel(CROSS)
    i ← i +1
```






# Properties

**Notation**

$G_s$: connected component of $s$

**Property 1**

$BFS(G, s)$ visits all the vertices and edges of $G_s$

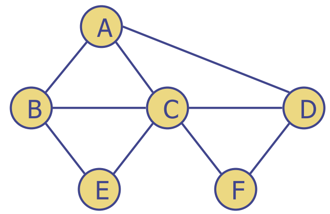
**Property 2**

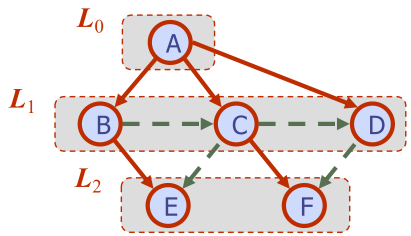
The discovery edges labeled by $BFS(G, s)$ form a spanning tree $T_s$ of $G_s$

**Property 3**

For each vertex $v$ in $L_i$

- The path of $T_s$ from $s$ to $v$ has $i$ edges
- Every path from $s$ to $v$ in $G_s$ has at least $i$ edges (i.e., find a shortest path)






# Analysis

- Setting/getting a vertex/edge label takes $O(1)$ time
- Each vertex is labeled twice
  - once as UNEXPLORED
  - once as VISITED
- Each edge is labeled twice
  - once as UNEXPLORED
  - once as DISCOVERY or CROSS
- Each vertex is inserted once into a sequence $L_i$
- Method incidentEdges is called once for each vertex
- BFS runs in $O(n + m)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$



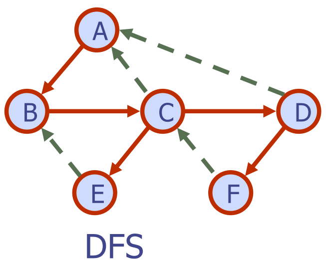
# Applications

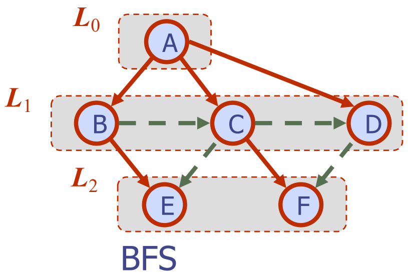
- Using the template method pattern, we can specialize the BFS traversal of a graph $G$ to solve the following problems in $O(n + m)$ time
  - Compute the connected components of $G$
  - Compute a spanning forest of $G$
  - Find a simple cycle in $G$, or report that $G$ is a forest
  - Given two vertices of $G$, find a path in $G$ between them with the minimum number of edges, or report that no such path exists

## DFS vs. BFS

| Applications | DFS | BFS |
| --- | --- | --- |
| Spanning forest, connected components, paths, cycles | √ | √ |
| Shortest paths | | √ |
| Biconnected components (how?) | √ | |

Biconnected components:
- Connected
- Even after removing any vertex the graph remains connected



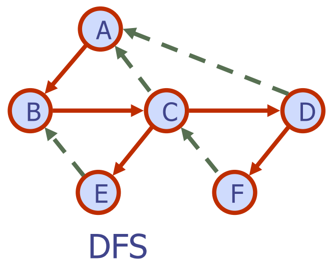



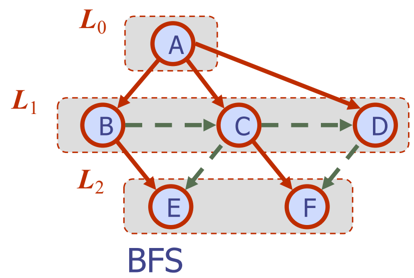



## DFS vs. BFS (cont.)

Back edge $(v,w)$
- $w$ is an ancestor of $v$ in the tree of discovery edges

Cross edge $(v,w)$
- $w$ is in the same level as $v$ or in the next level






# Questions?