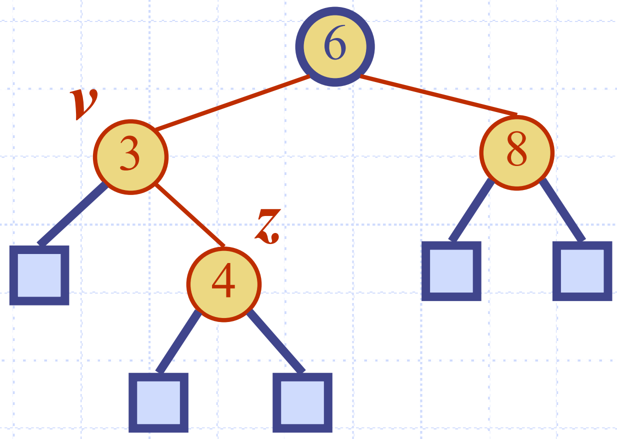
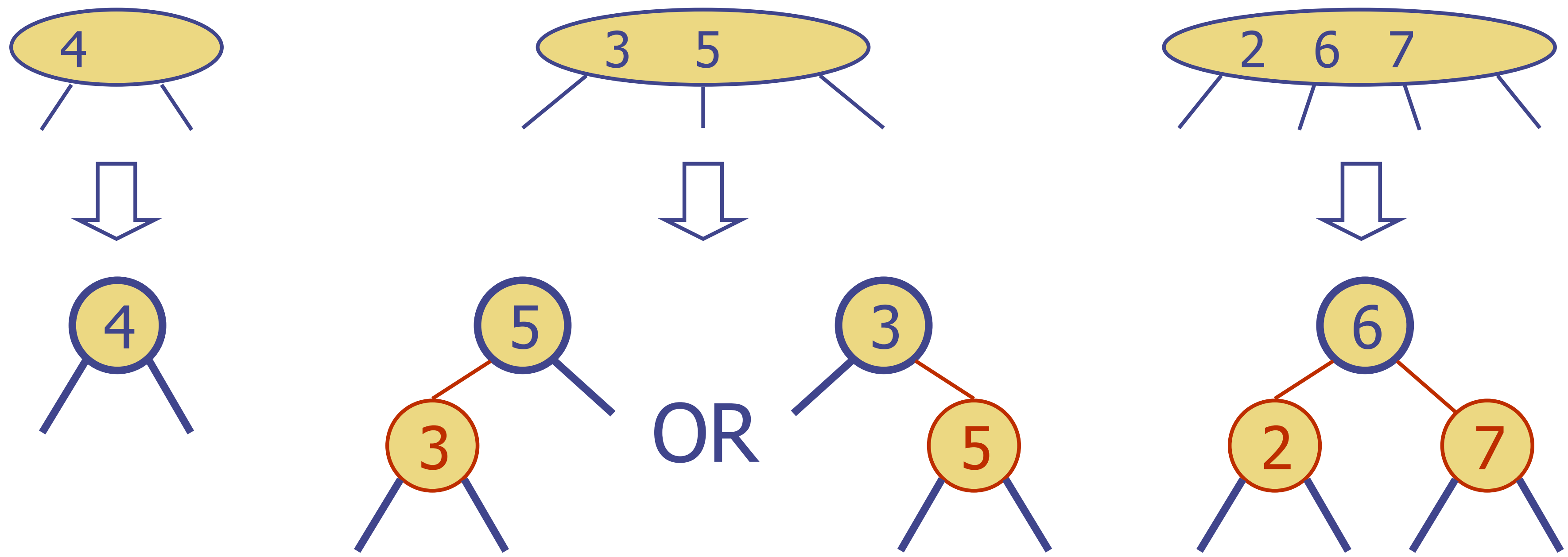
Red-Black Trees



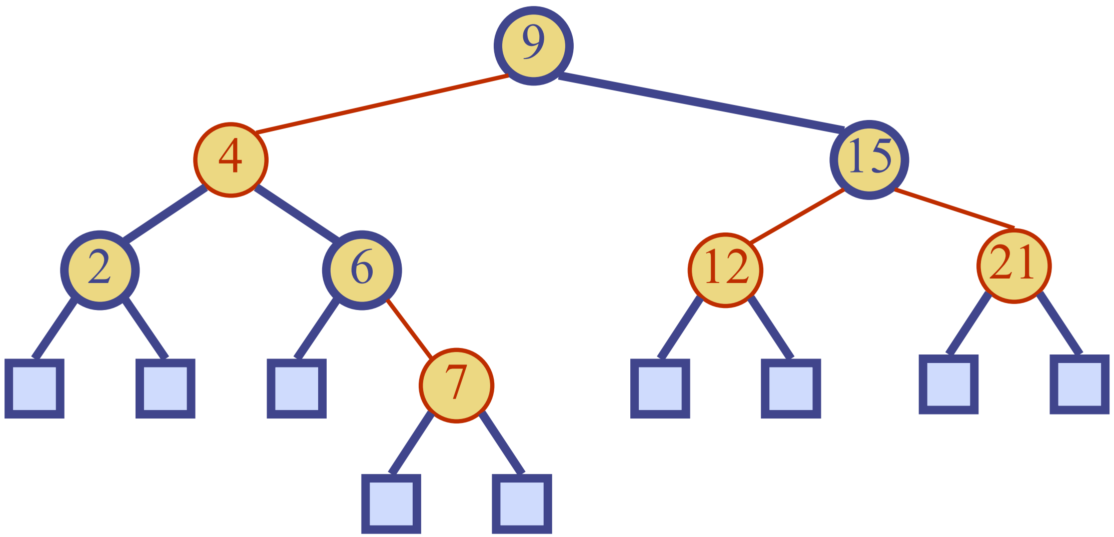
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored **red** or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type

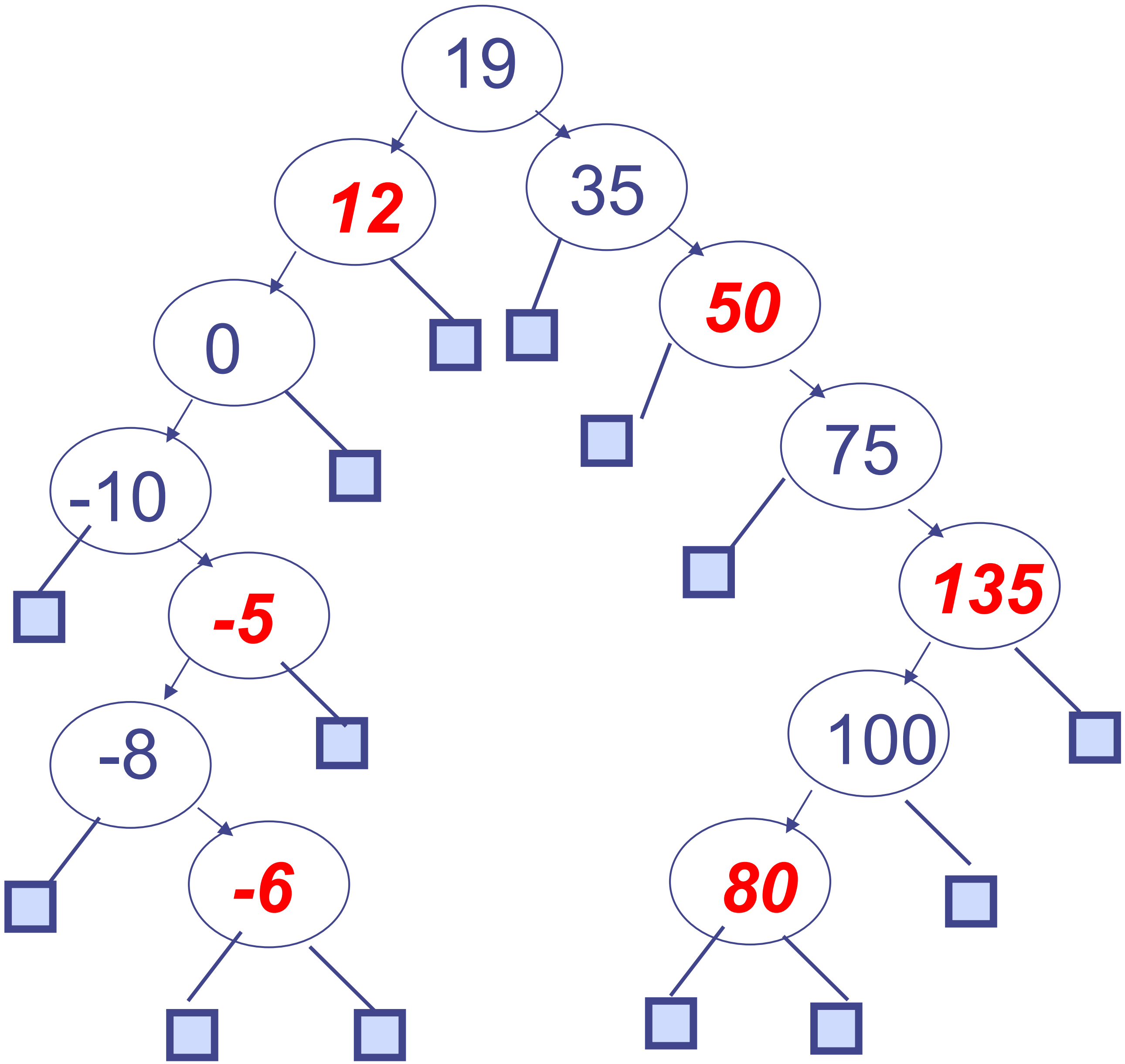


Red-Black Trees

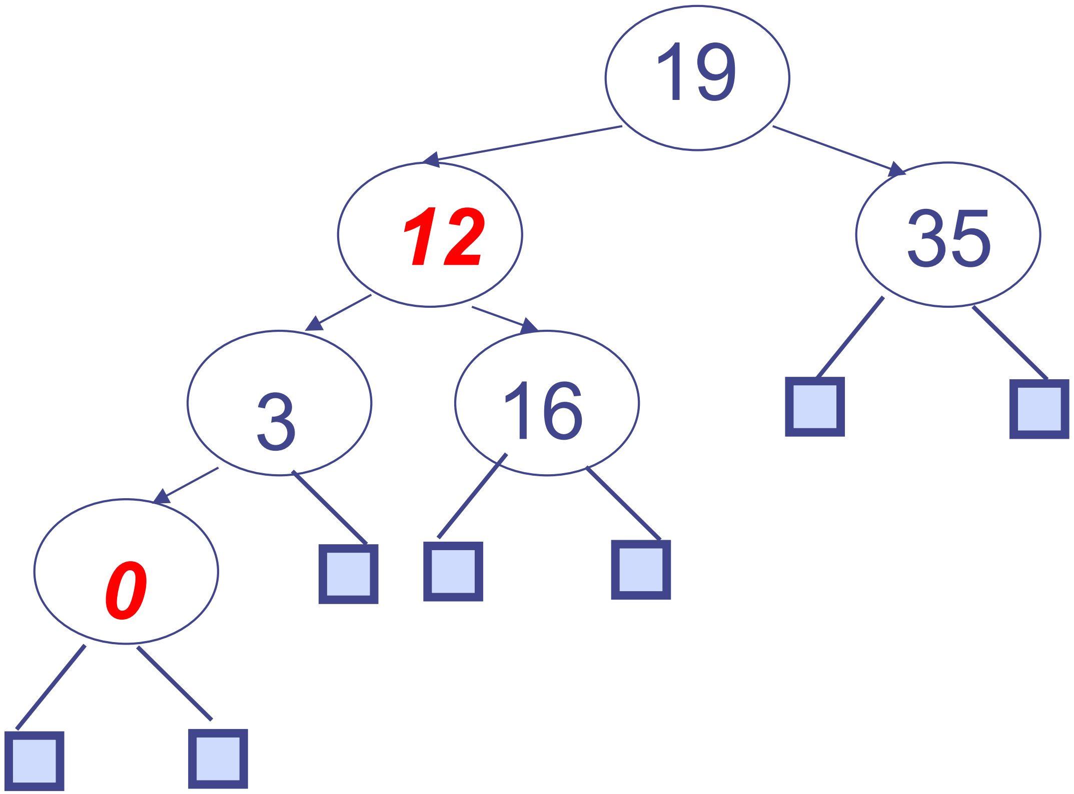
- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black (red rule)
  - Depth Property: all the leaves have the same black depth (path rule)
  - (Question) How is balancing enforced here?



Red Black Tree?



Red Back Tree?



What if we attach a child to node 0?

Implications

- Root Property: the root is black
- External Property: every leaf is black
- Internal Property: the children of a red node are black (red rule)
- Depth Property: all the leaves have the same black depth (path rule)

- 1. If a red node has any children, it must have two children and they must be black
  - Why? Depth property

- 2. If a black node has only one “real” child then it must be a ”last” red node
  - If the child is black?
  - If the child is not the last red?

- (Question) How is balancing enforced in R-B tree?

Intuition about “rough balancing”

- The longest path <= 2 * the shortest path
  - Rough balancing → guarantees *log(n)* height

- Why?
  - From “red rule” and “path rule”
    shortest path = only black nodes
    longest path = inserting a red node between two black nodes

Root Property: the root is black
External Property: every leaf is black
Internal Property: the children of a red node are black (red rule)
Depth Property: all the leaves have the same black depth (path rule)

Height of a Red-Black Tree

- Theorem: A red-black tree storing $\boldsymbol{n}$ entries has height $\boldsymbol{O}(\log \boldsymbol{n})$

  Proof:
  - Omitted

- The search algorithm for a binary search tree is the same as that for a binary search tree

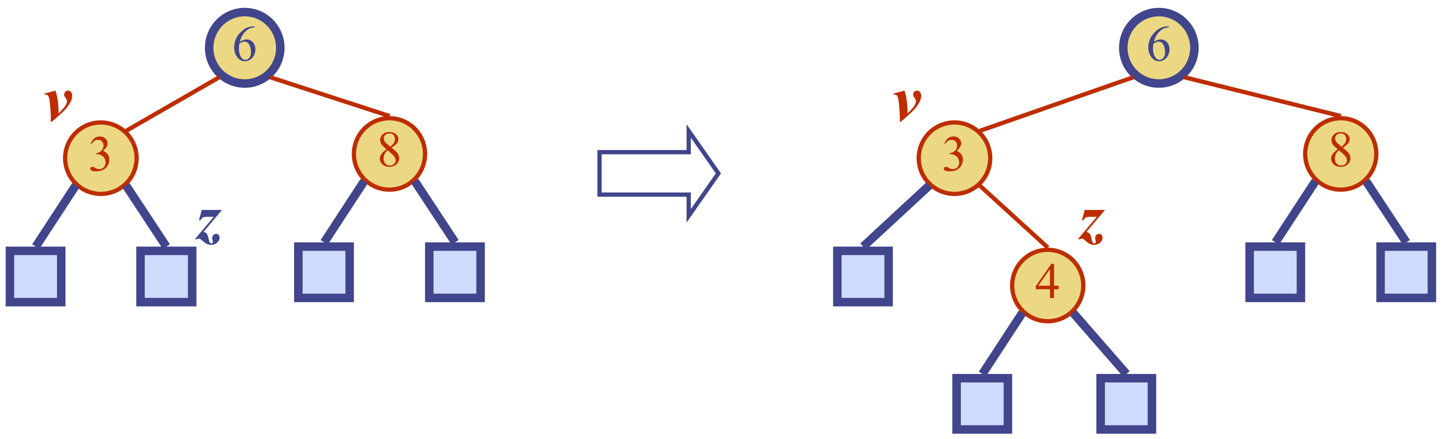
- By the above theorem, searching in a red-black tree takes $\boldsymbol{O}(\log \boldsymbol{n})$ time

Insertion

Insertion

- To perform operation put(k, o), we execute the insertion algorithm for binary search trees and color red the newly inserted node z unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent v of z is black, we also preserve the internal property and we are done
  - Else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
  - Goal: Removing double read without breaking the depth property

- Example where the insertion of 4 causes a double red:

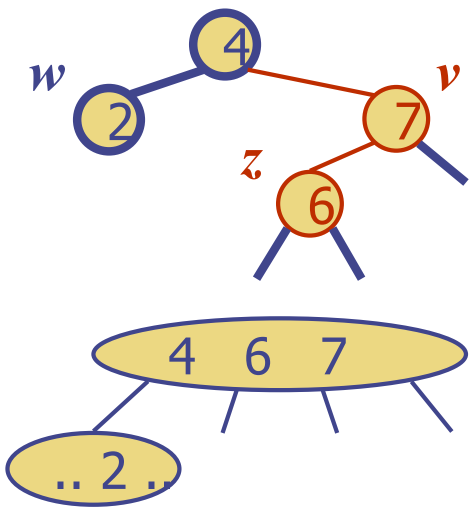


Remedying a Double Red

- Consider a double red with child z and parent v, and let w be the sibling of v

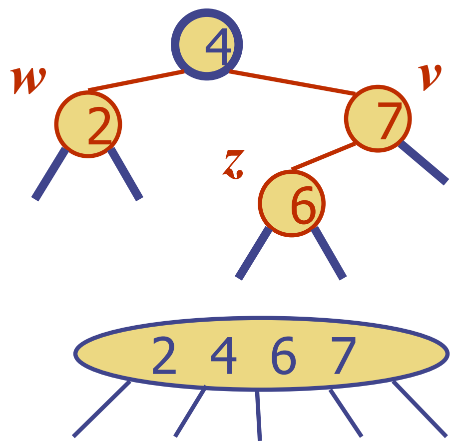
Case 1: w is black

- Viewpoint
  The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



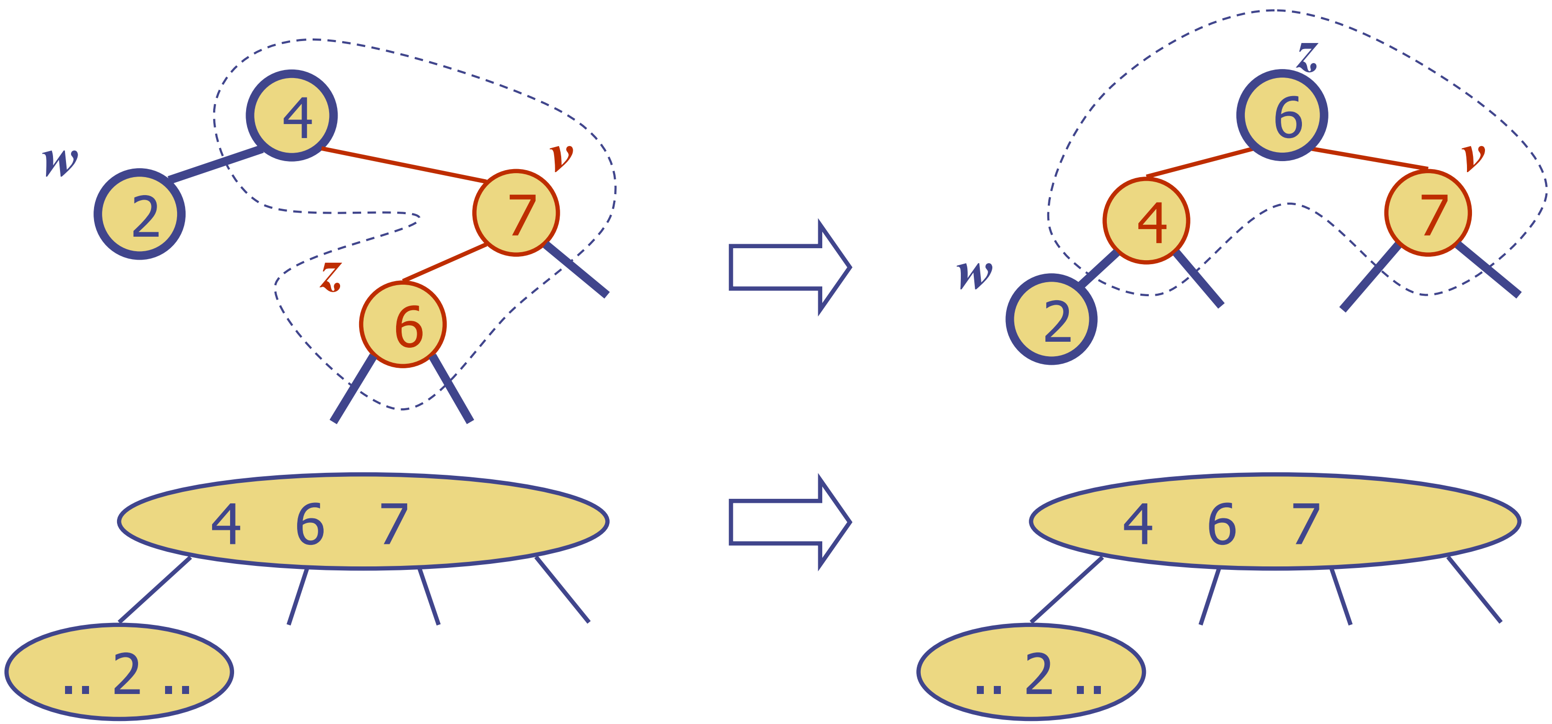
Case 2: w is red

- Viewpoint
  The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



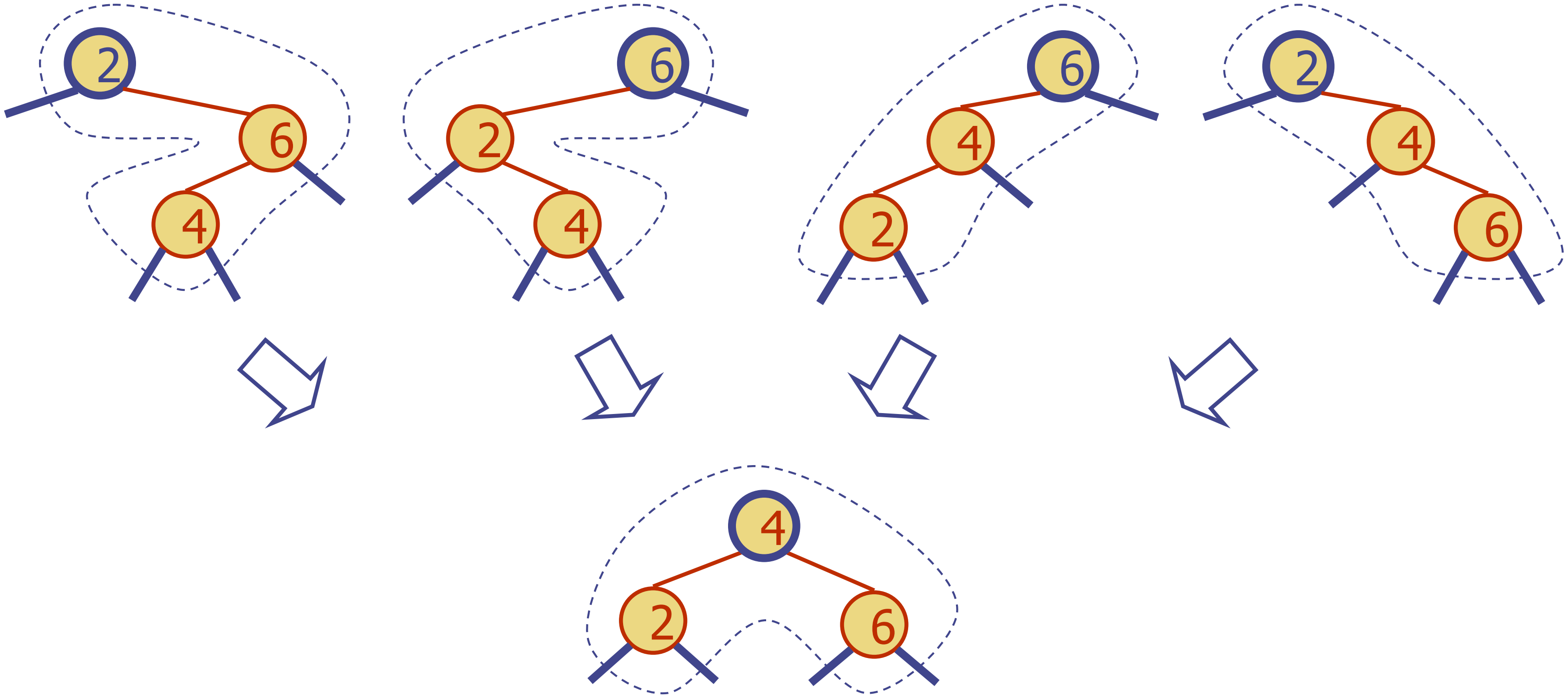
Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



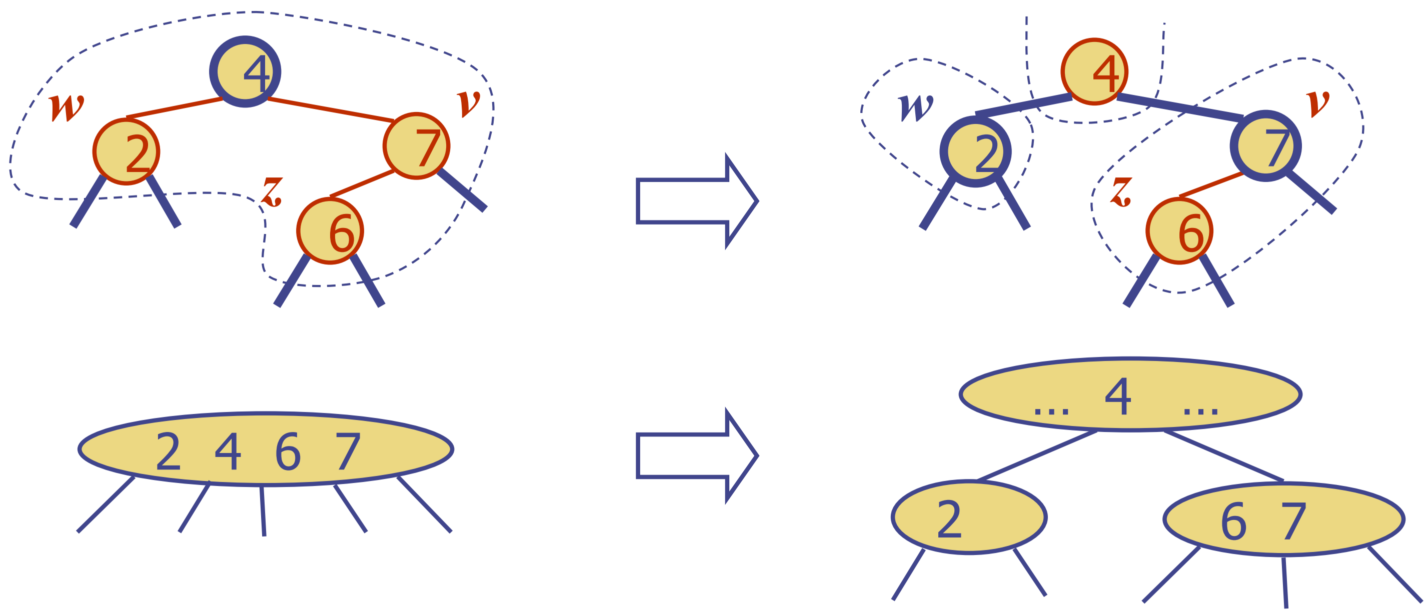
Restructuring (cont.)

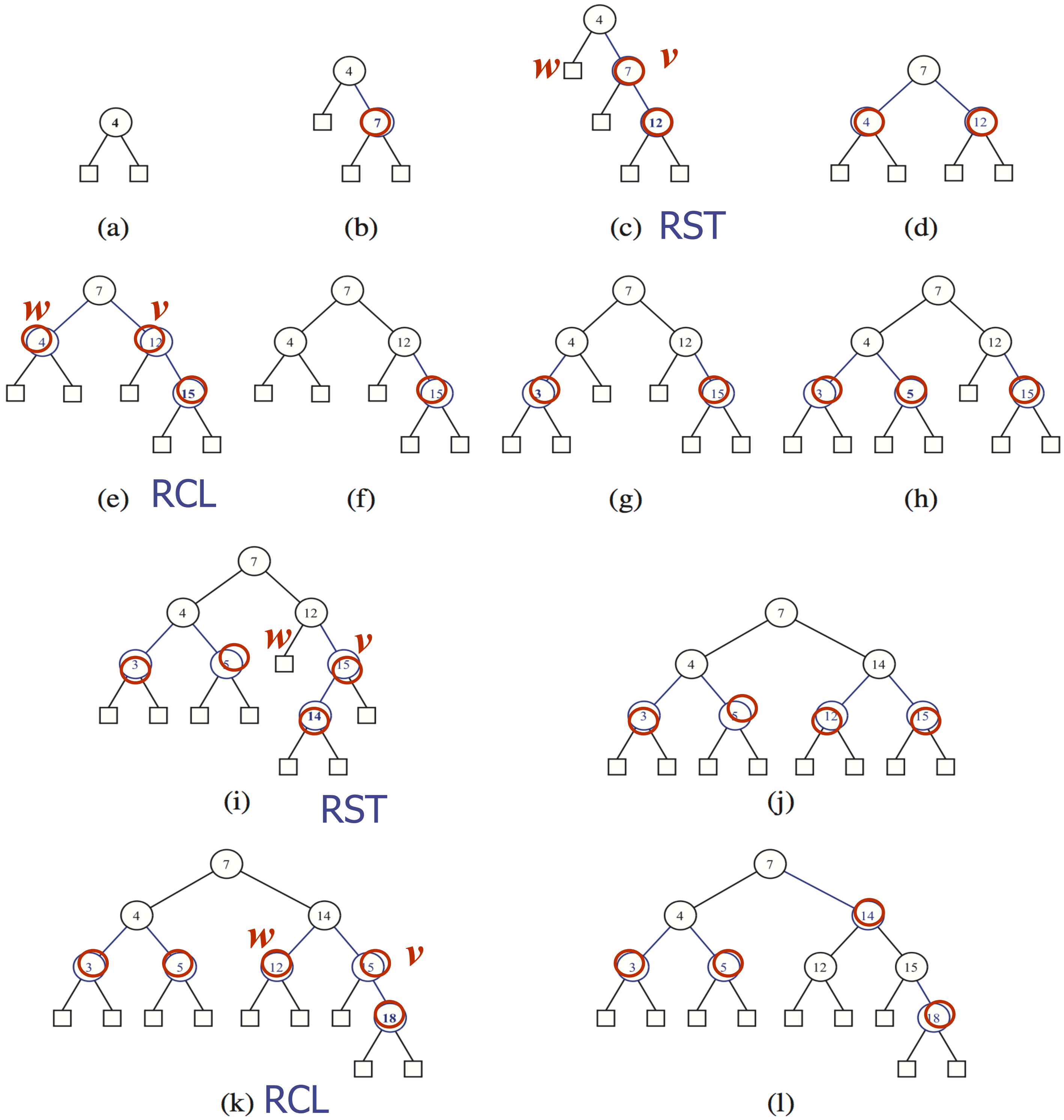
- There are four restructuring configurations depending on whether the double red nodes are left or right children



Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent v and its sibling w become black and the grandparent u becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent u




4
(a)
4
7
(b)
4
w
7
v
12
(c) RST
7
4
12
(d)
7
w
4
v
12
15
(e) RCL
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(f)
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(g)
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(j)
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(k) RCL
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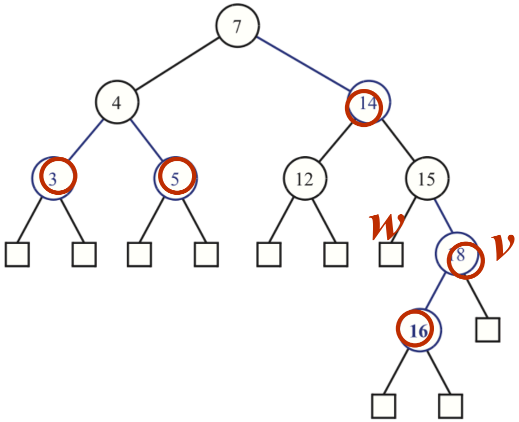
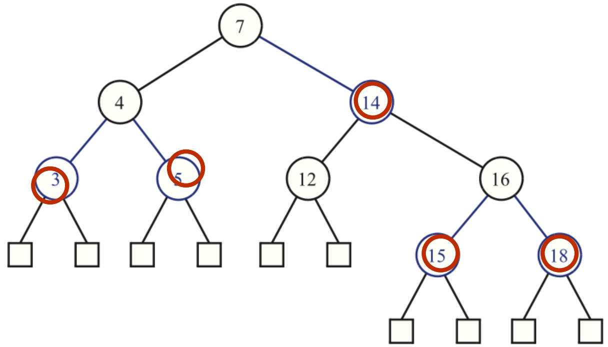
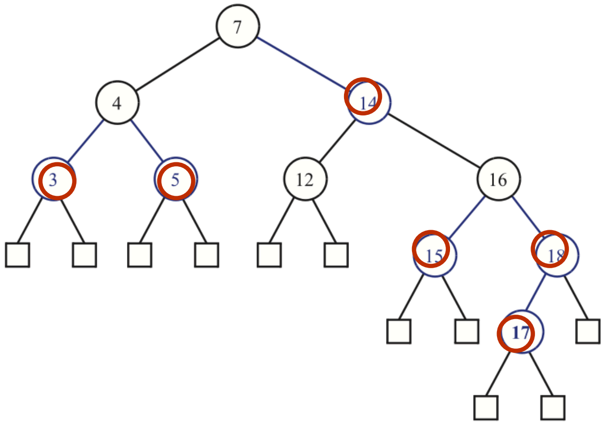


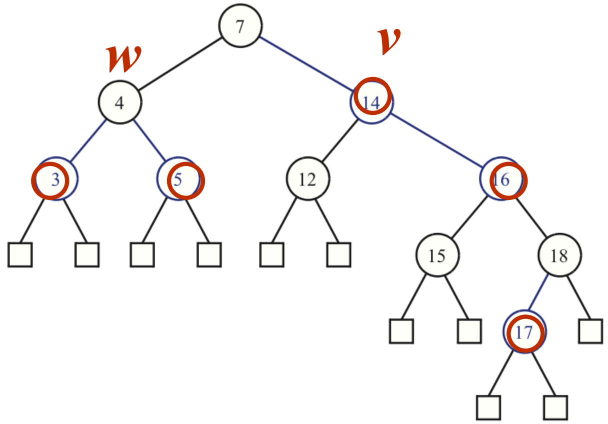
(m) RST



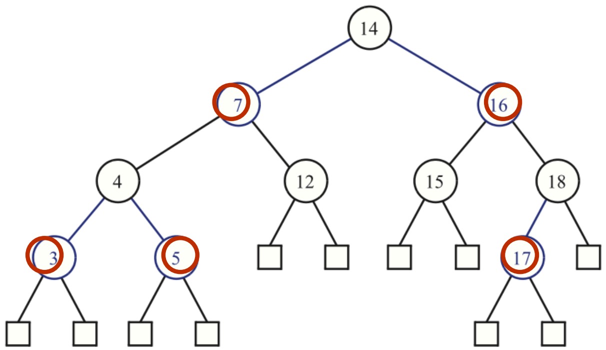
(n)



(o) RCL



(p) RST



(q)

Analysis of Insertion

Algorithm *put*(k, o)

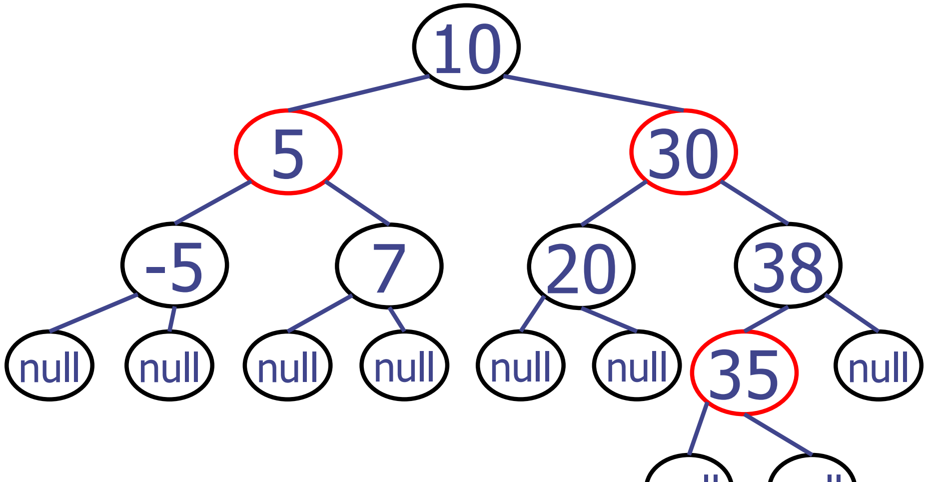
1. We search for key k to locate the insertion node z
2. We add the new entry (k, o) at node z and color z red
3. **while** *doubleRed*(z)
   if *isBlack*(*sibling*(*parent*(z)))
   $z \leftarrow$ *restructure*(z)
   return
   else { *sibling*(*parent*(z)) is red }
   $z \leftarrow$ *recolor*(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
  - $O(\log n)$ recolorings, each taking $O(1)$ time, and
  - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

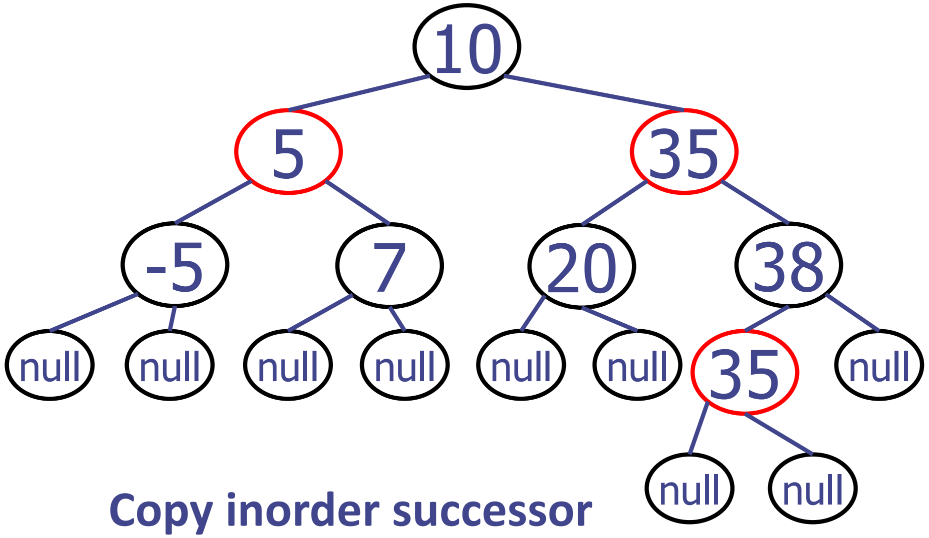
RB-Tree: Deletion

Deletion: Example 1

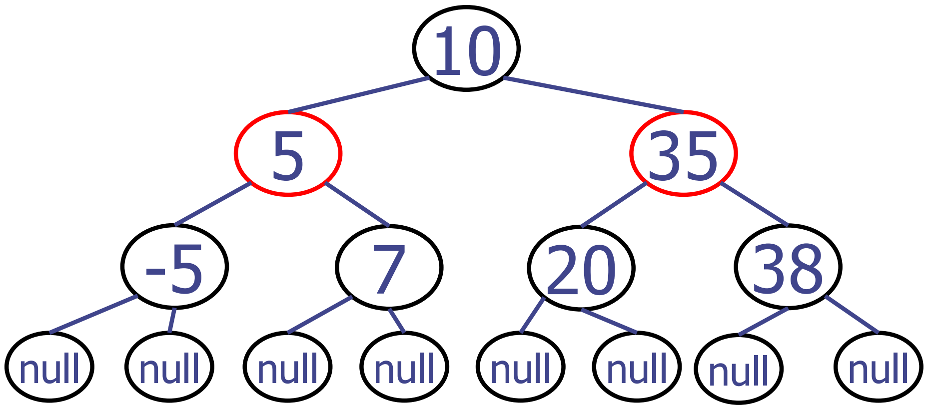
- To perform operation erase(k), we first execute the deletion algorithm for binary search trees



delete 30



Copy inorder successor

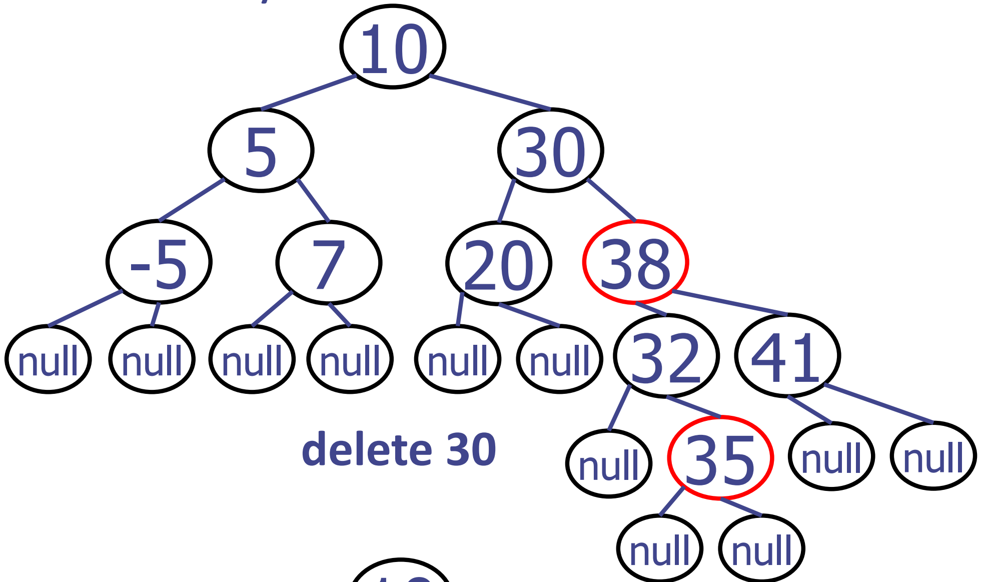


Just delete the copied 35, and color the remaining node in black. Then, we are done.

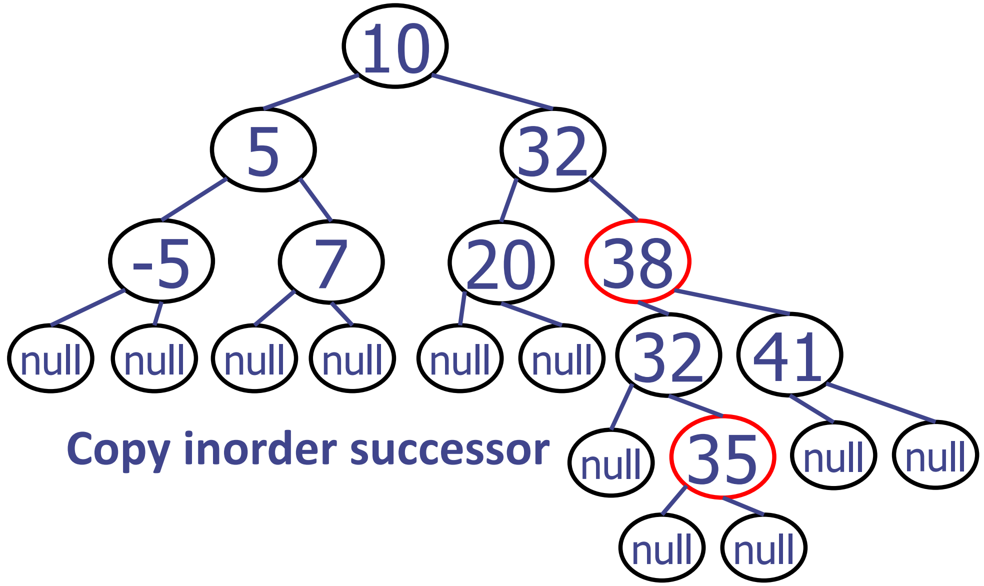
Implication:
<u>**If the node to be deleted is red**</u>, removing it is fine

Deletion: Example 2

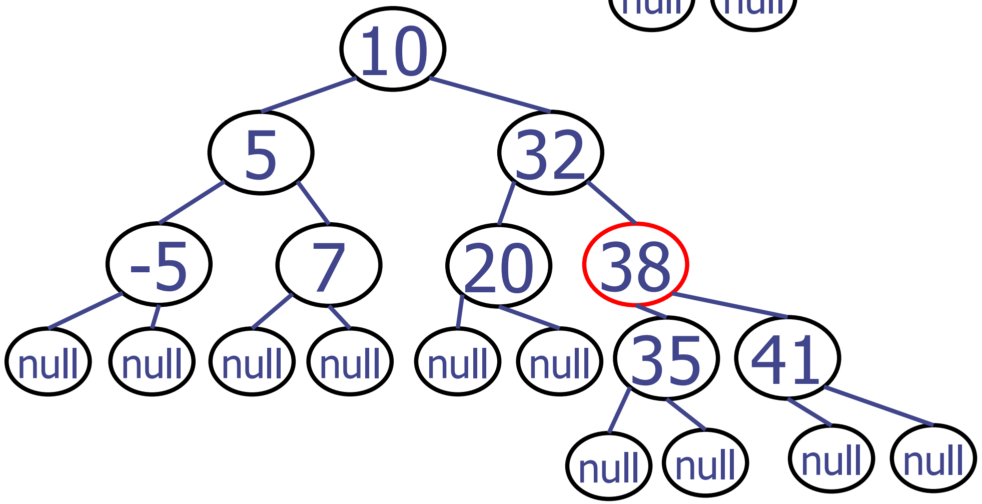
- To perform operation erase(k), we first execute the deletion algorithm for binary search trees



delete 30



Copy inorder successor

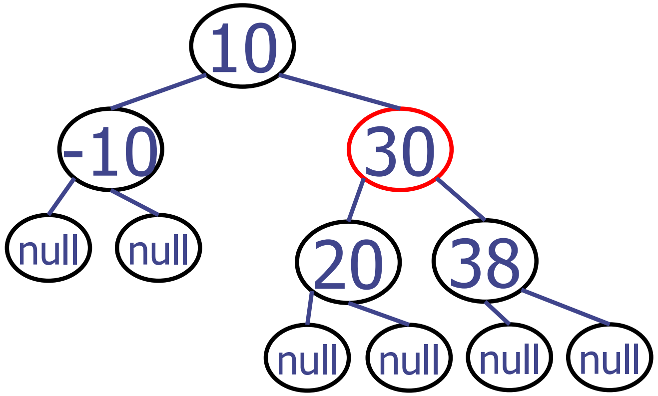


Just delete the copied 32, and color 35 with black.

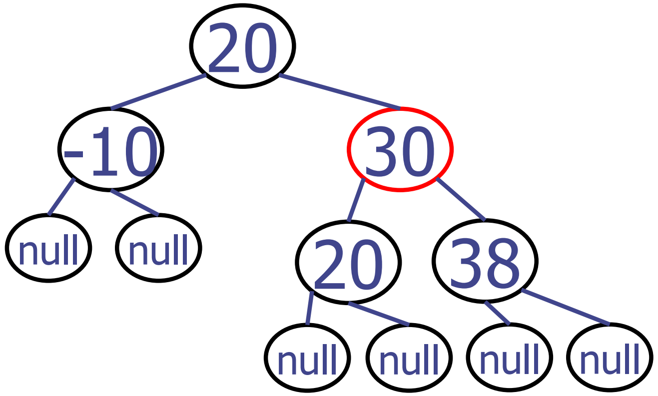
Implication: **For a node (with a red child) to be deleted**, delete it and change the red child's color.
(35: -1 first and +1 second. So no change)

Deletion: Example 3

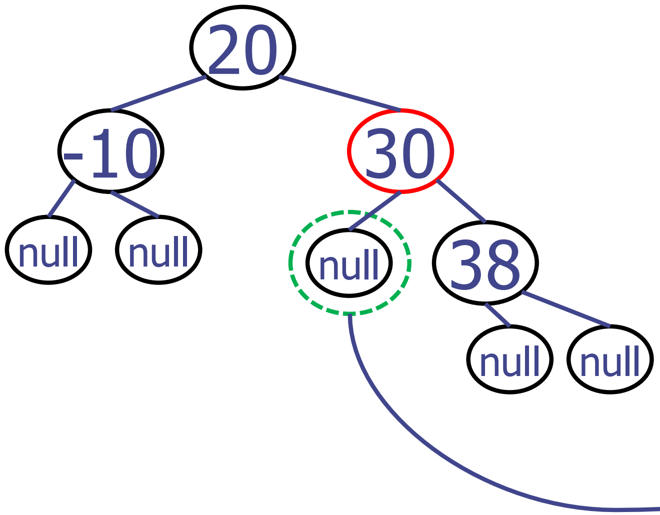
- What about deleting **<u>a node with a black child</u>**?



Delete 10



Copy inorder successor



Delete 20.

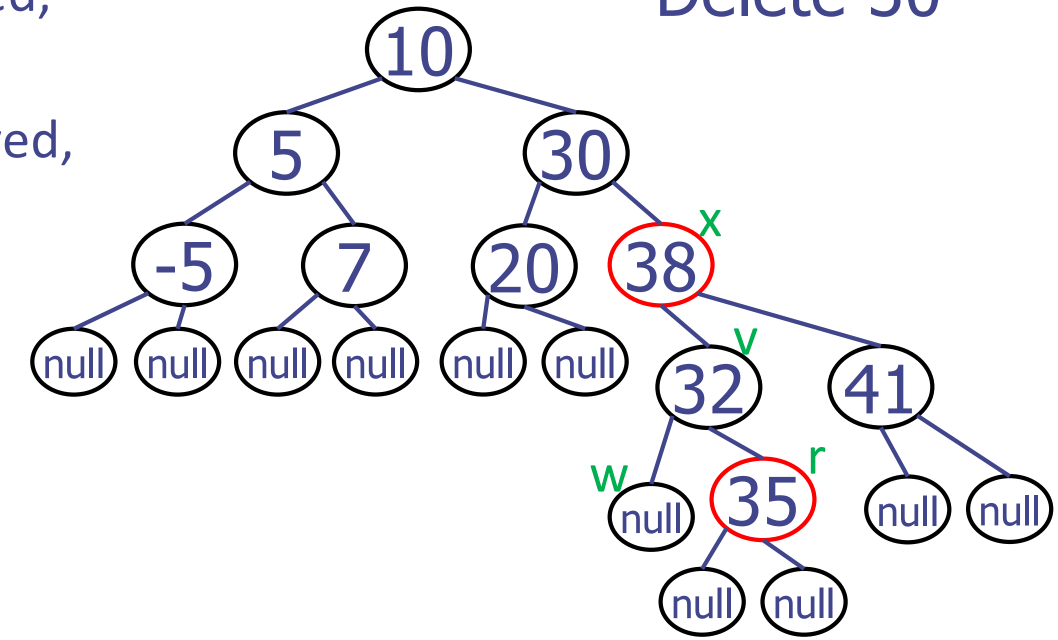
Problem: A path of only 2 blacks

Regard this as "double black nodes"

Deletion

- To perform operation erase(k), we first execute the deletion algorithm for binary search trees
  - Enough to consider the removal of an entry at a node with an external child (To remove a node with both internal children, we first copy the inorder successor, and then …)
- Notations
  - v : the internal node removed,
    - "myself"
  - w : the external node removed,
    - "my lonely child"
  - r : the sibling of w
    - "my other child"
  - x : the parent of v
    - "my father"

Delete 30



Questions

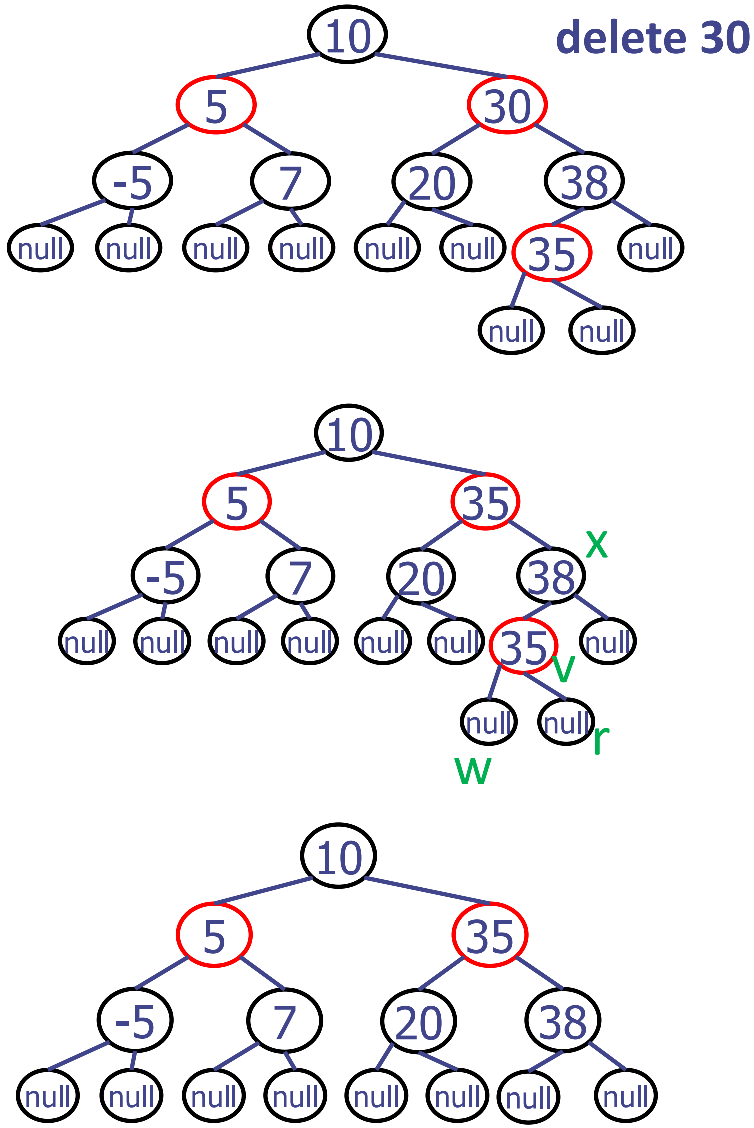
- How to handle “double black nodes”
- Are there some cases in handling those? Yes
- Are you ready for ”cases”?
- It’s really, really complex, but if you concentrate, then you can follow it.

Deletion: Algorithm Overview (1)

First, remove v and w, and make r a child of x

If either of v or r was red, we color r black and we are done (Examples 1 and 2)

Else (v and r were both black) we color r ***double black***, which is a violation of the internal property requiring a reorganization of the tree (Examples 3)

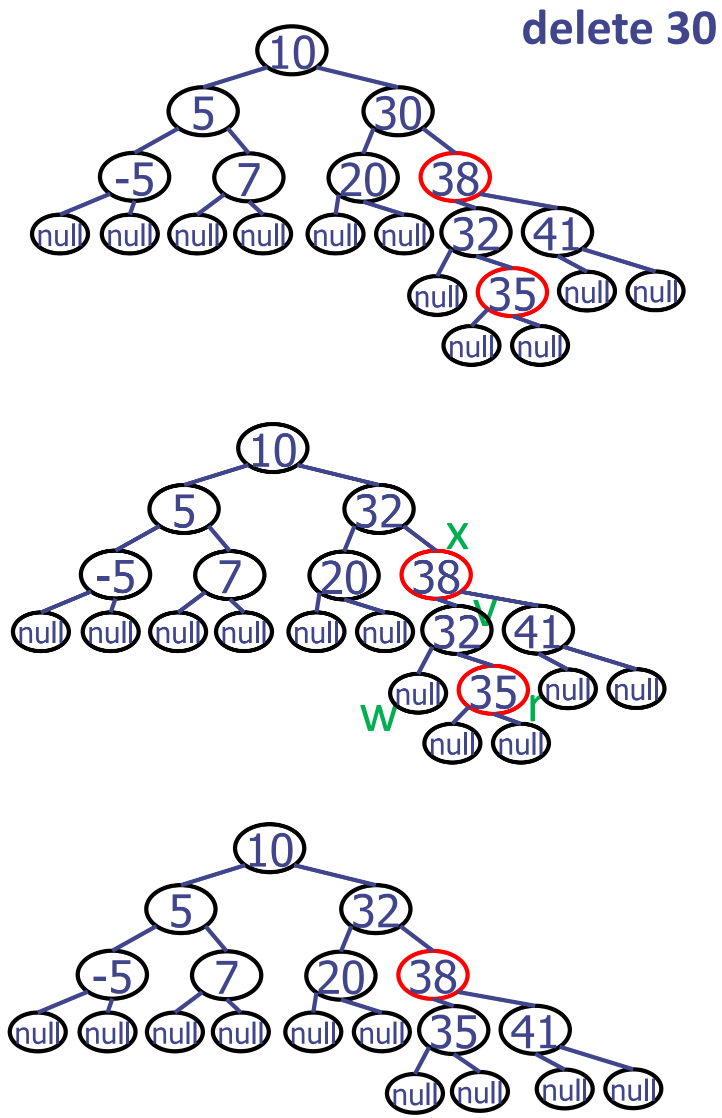


Deletion: Algorithm Overview (2)

First, remove v and w, and make r a child of x

If either of v or r was red, we color r black and we are done (Examples 1 and 2)

Else (v and r were both black) we color r ***double black***, which is a violation of the internal property requiring a reorganization of the tree (Examples 3)



Deletion: Algorithm Overview (2)

First, remove v and w, and make r a child of x

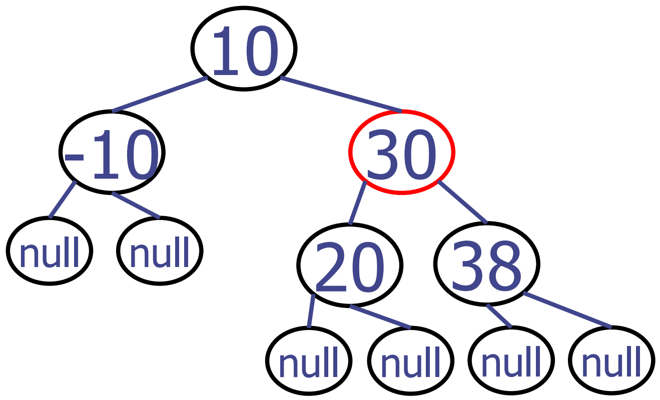
If either of $\boldsymbol{v}$ or $\boldsymbol{r}$ was red, we color $\boldsymbol{r}$ black and we are done (Examples 1 and 2) *(Let's call this Case 0)*

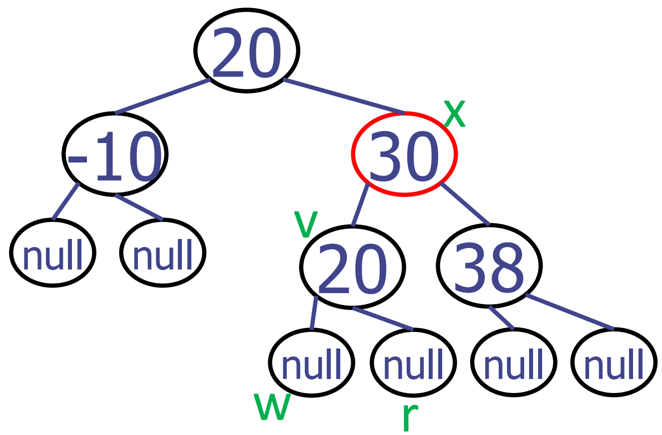
Else ($\boldsymbol{v}$ and $\boldsymbol{r}$ were both black) we color $\boldsymbol{r}$ ***double black***, which is a violation of the internal property requiring a reorganization of the tree (Examples 3)

- Notations after removing v and w
  - y: sibling of r
  - z: child of y
- We now divide the cases, depending of the color of y and z

Recall: Example 3. Notations again!

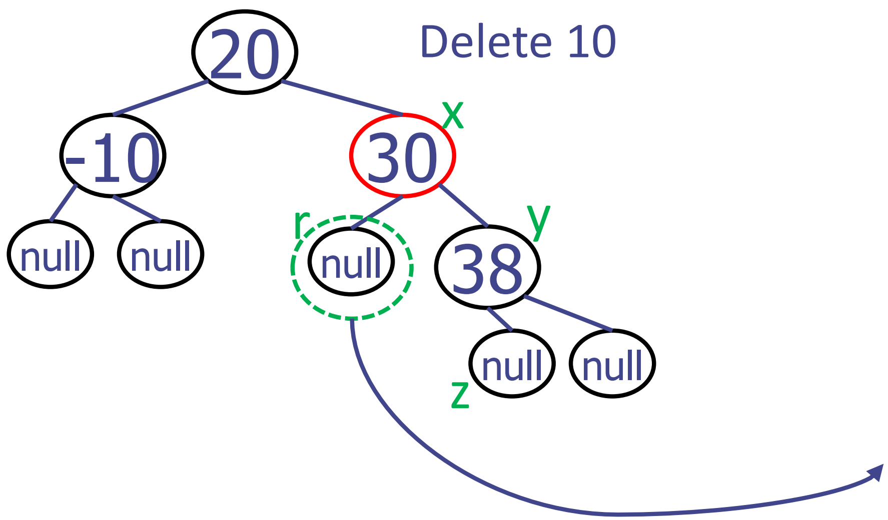
- What about deleting a node with a black child?





Delete 10

Copy inorder successor



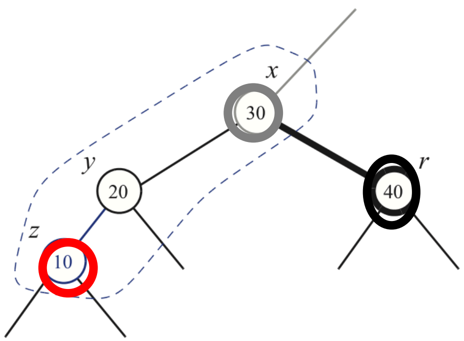
Delete 20.

Problem: A path of only 2 blacks

Regard this as “double black nodes”

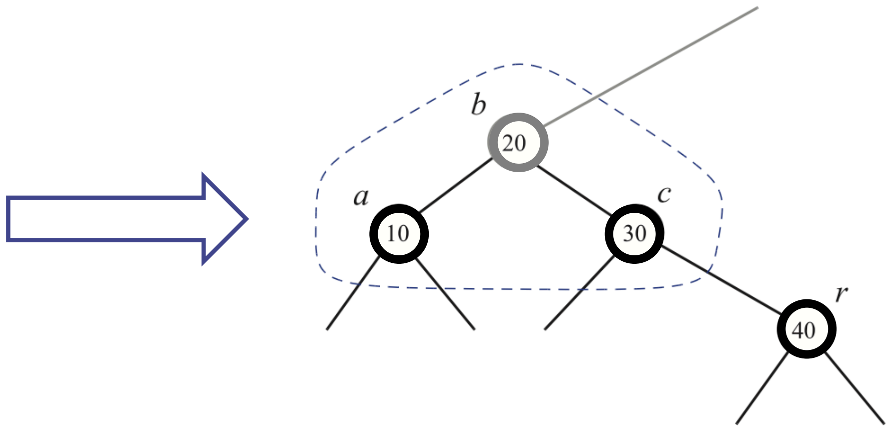
Handling Double Black Nodes: Case 1

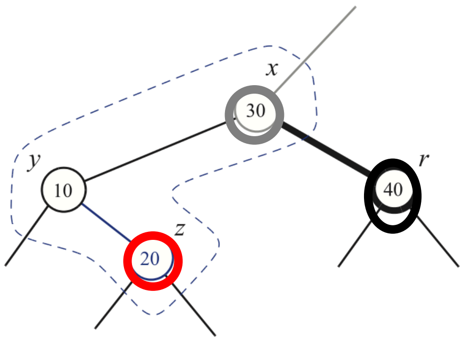
- Case 1: The sibling y of r is black, and has a red child z
  - We perform a restructuring, and we are done



z is the left child

(a)





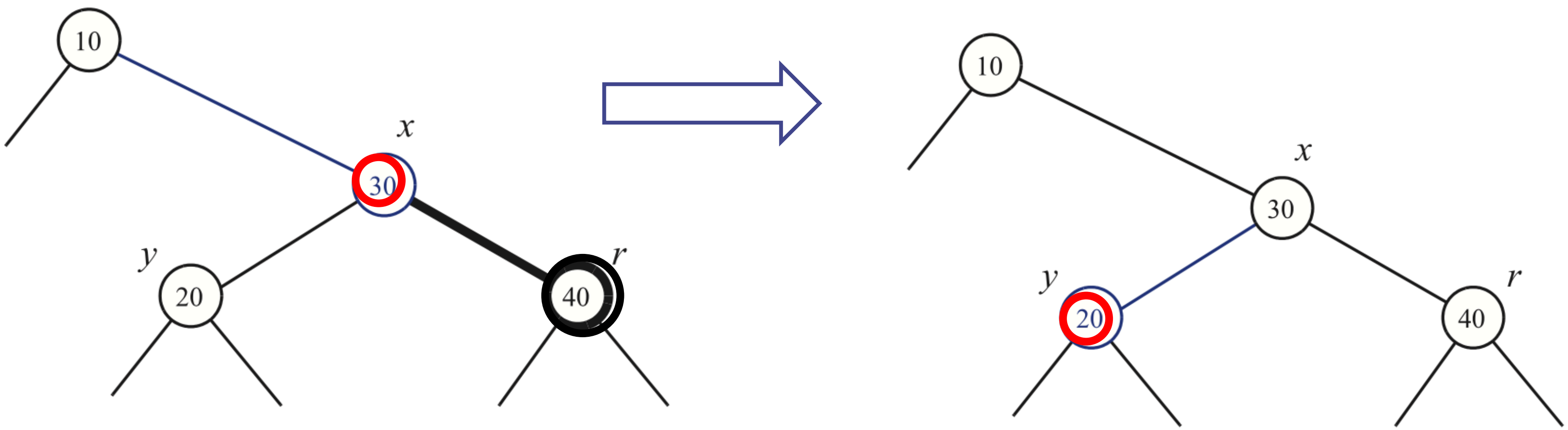
z is the right child

(b)

Double black node solved?

Handling Double Black Nodes: Case 2

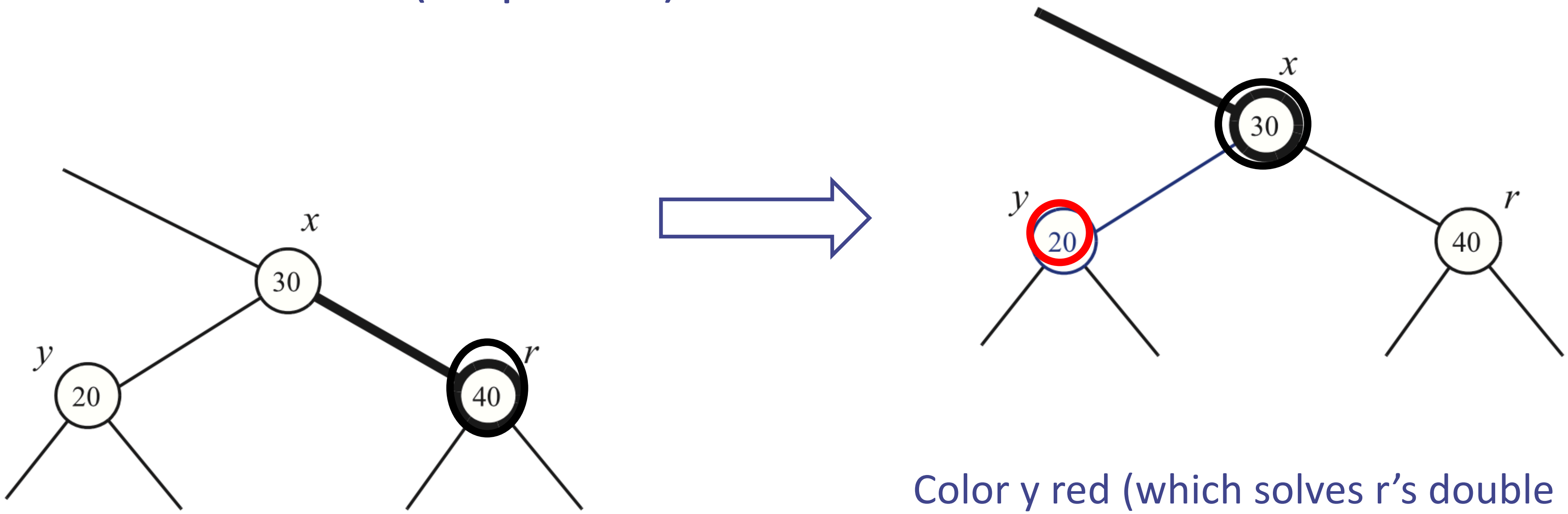
- Case 2: The sibling y of r is black, and y's both children are black
  - We perform a recoloring
  - Case 2-1: x (r's parent) is red



Color x black and color y red

Handling Double Black Nodes: Case 2

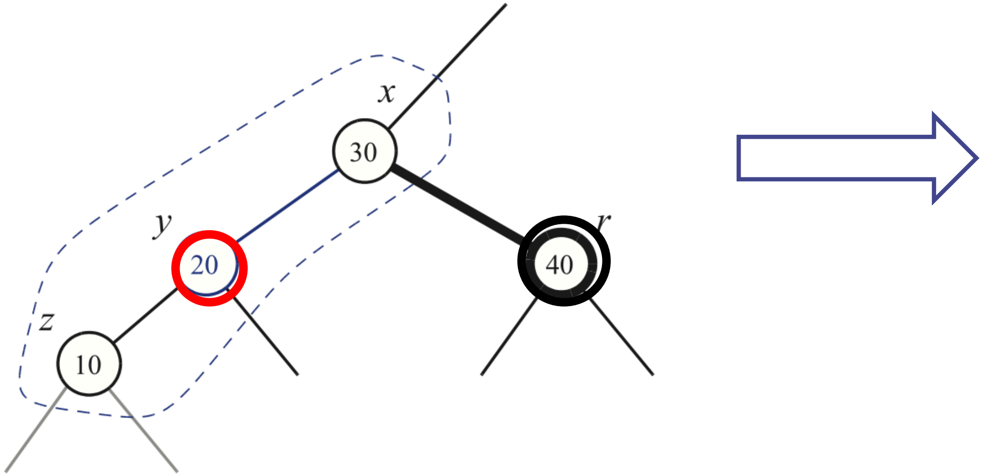
- Case 2: The sibling y of r is black, and y’s both children are black
  - We perform a recoloring
  - Case 2-2: x (r’s parent) is black

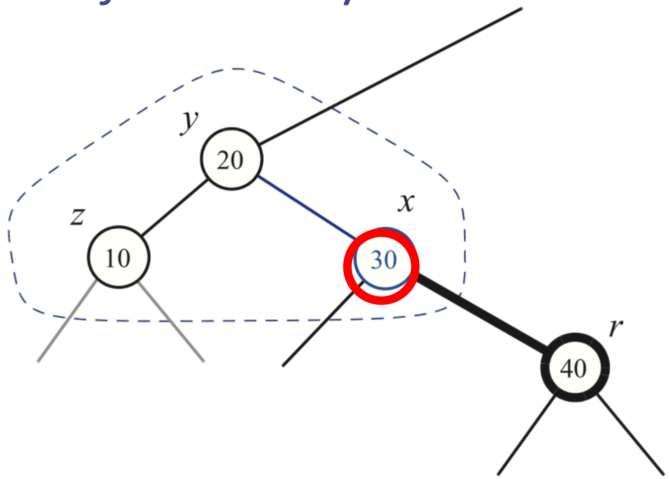


Color y red (which solves r’s double black), and make x “double black” (propagates the double black up), then reconsider the cases for x

Handling Double Black Nodes: Case 3

- Case 3: The sibling y of r is red
  - We perform adjustment
    - If y is the *right* child of x, then let z be the *right* child of y
    - If y is the *left* child of x, then let z be the *left* child of y
  - Case 3-1: z is the left child of y





Perform restructuring
Make y be the parent of x
Color y black and x red
(double black not yet solved)
→ The sibling of r is black (why?)
→ Case 1 or Case 2 applies

- Case 3-2: z is the right child of y → Similarly, we apply

Double Black Node Handling: Summary

- The algorithm for remedying a double black node r with sibling y considers three cases

  Case 1: y is black and has a red child

  - We perform a restructuring, and we are done

  Case 2: y is black and its children are both black

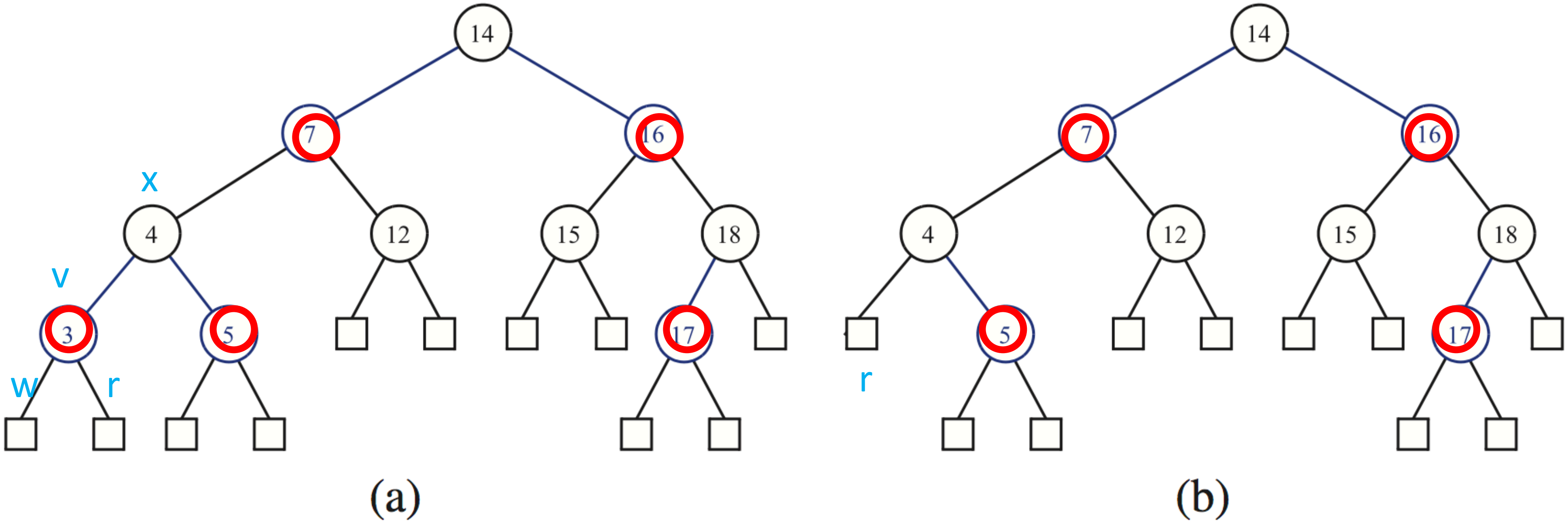
  - We perform a recoloring, which may propagate up the double black violation

  Case 3: y is red

  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies

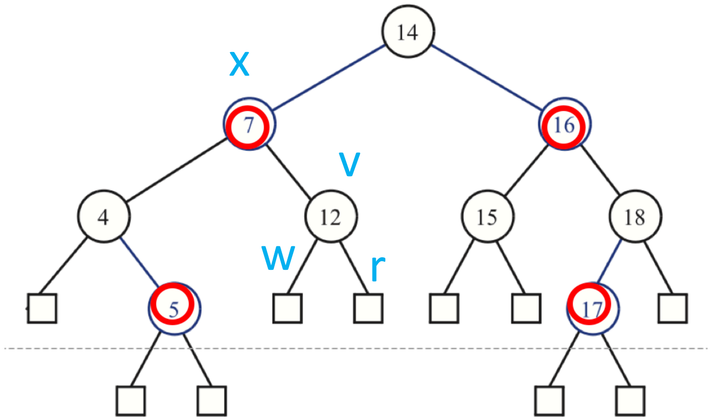
- Deletion in a red-black tree takes $O(\log n)$ time

Example: Remove 3



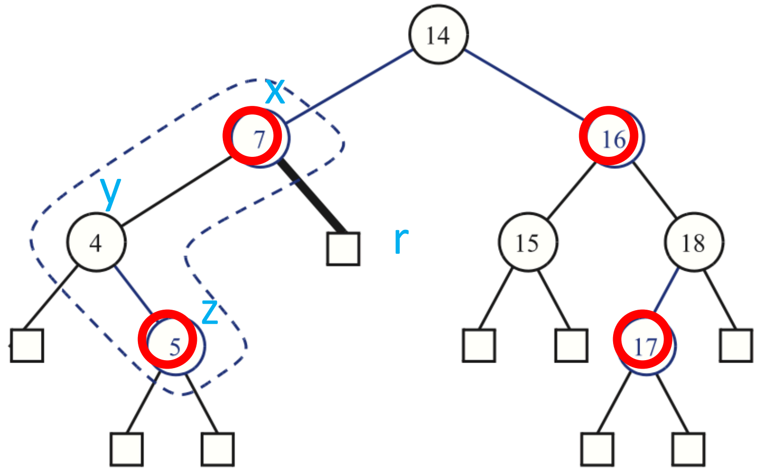
- v is red → Case 0 (either v or r is red)
- Remove v and w and color r black

Example: Remove 12

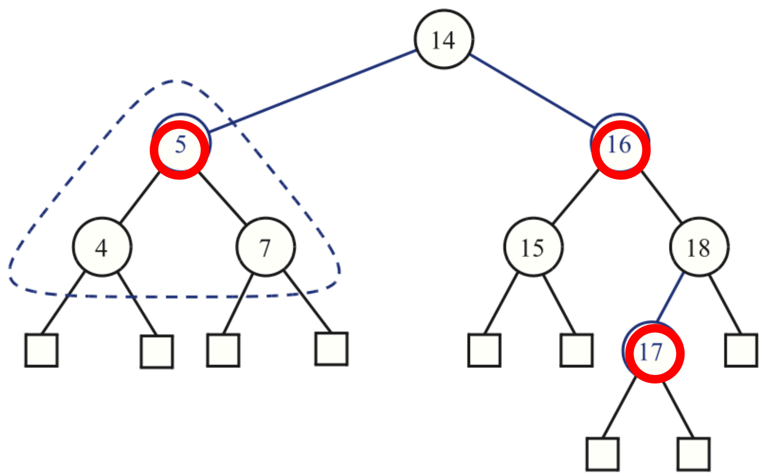


(b)

- None of v and r is red → Not Case 0
- y is black, which has red child
  - → Case 1, restructuring

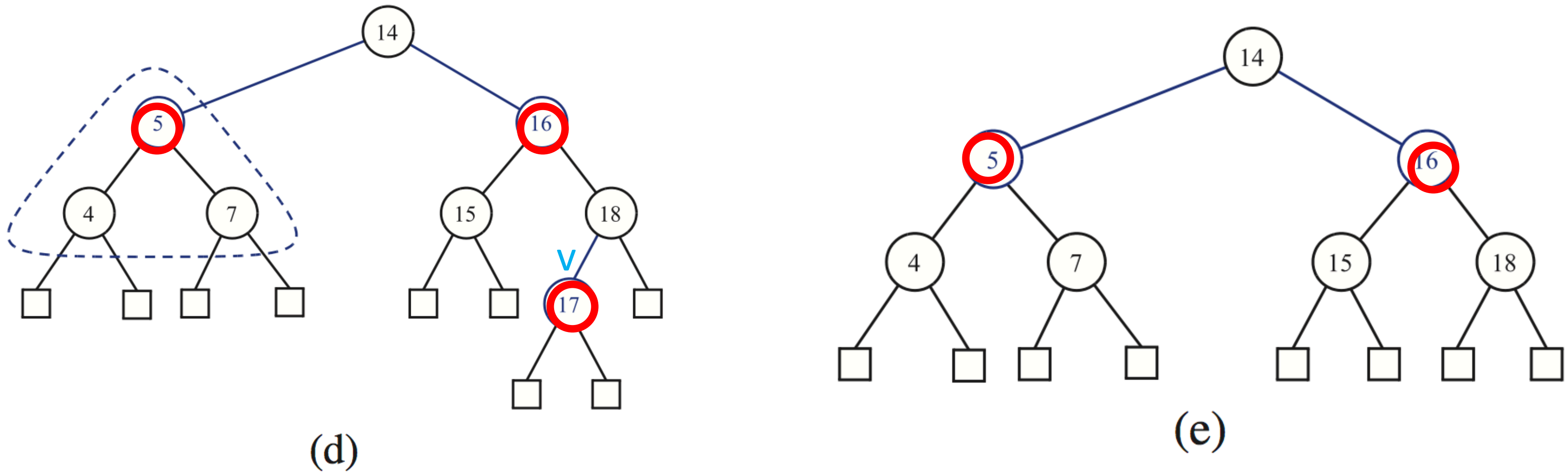


(c)



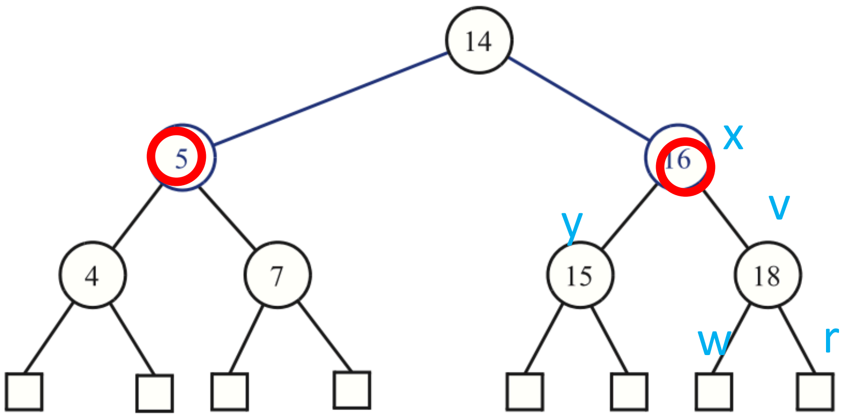
(d)

Example: Remove 17

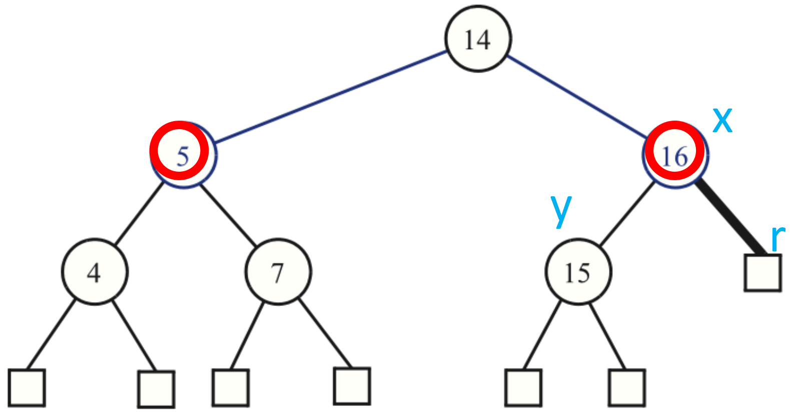


- v is red →Case 0

Example: Remove 18

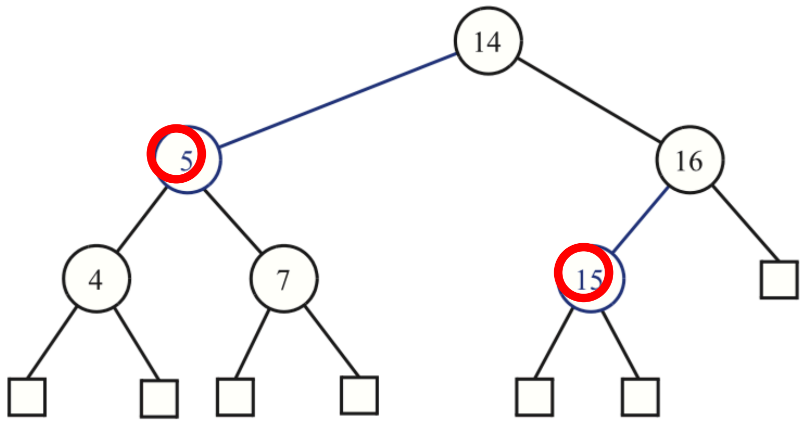


(e)



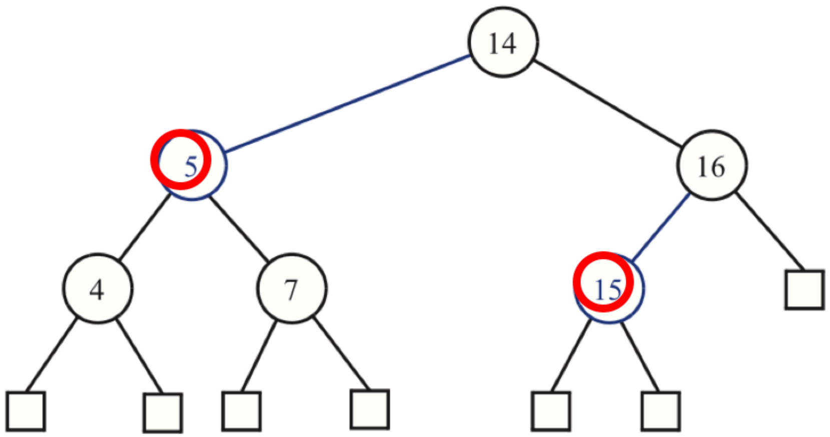
(f)

- None of v and r is red → Not Case 0
- y is black, having both black children →Case 2
  - x is red →Case 2-1, recoloring between x and y

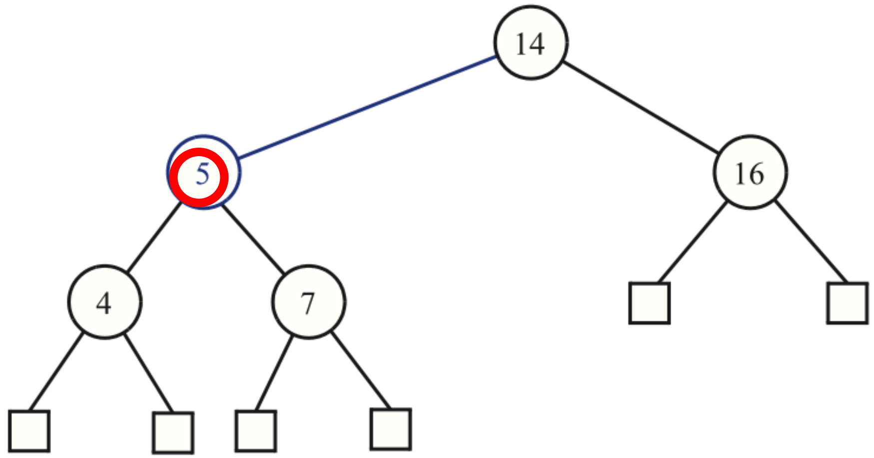


(g)

Example: Remove 15



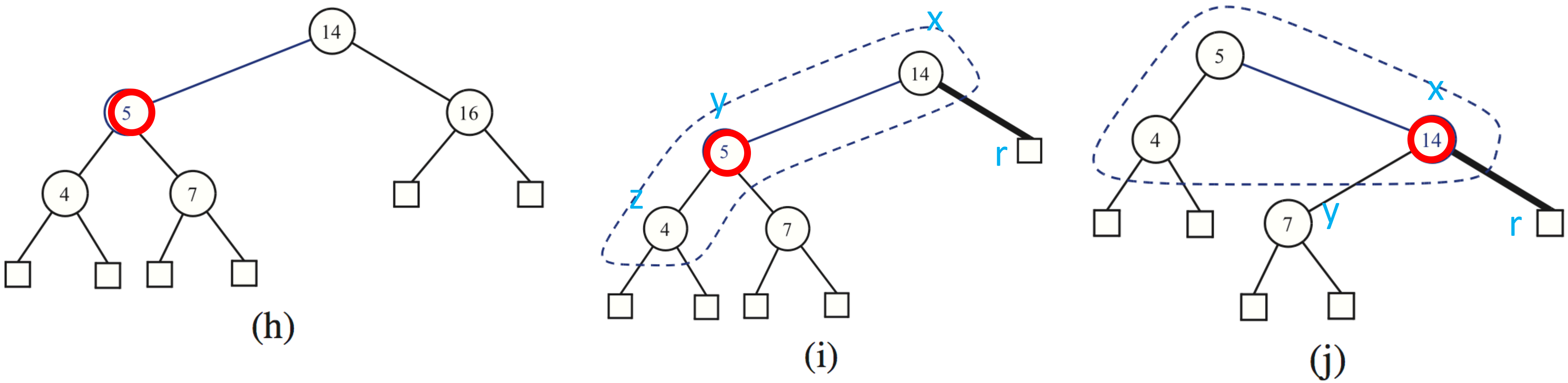
(g)



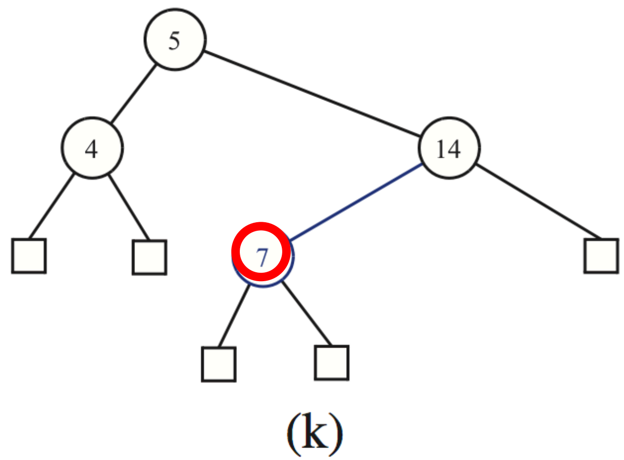
(h)

- Case 0 (now you know, right?)

Example: Remove 16



- y is red → Case 3
- y is the left child of x, thus z is node 4 (left child of y) → Case 3-1
- Adjustment → node 14 becomes double black → new y (sibling of x)
- y has both black children, and x is red
  - → Case 2-1, recoloring, then we're done



Questions?