Chapter 8

Epidemics on General Graphs

2016.11.14

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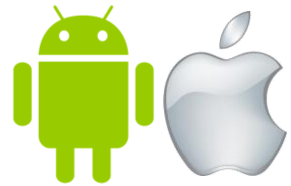
Spread over Social Network

- Diffusion by interaction among individuals

Virus

Rumor

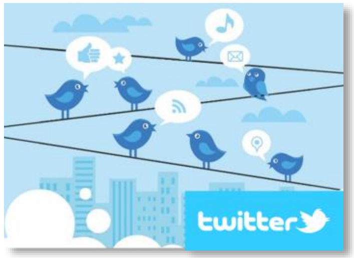
OS



Smartphone

Political Party

- New advertising opportunity via social network



Online Social Network Services

Survey: 1,713 companies in USA [Salseforce 2015]

72% will increase cost for social marketing
78% have a dedicated social media team
70% believe social marketing is core

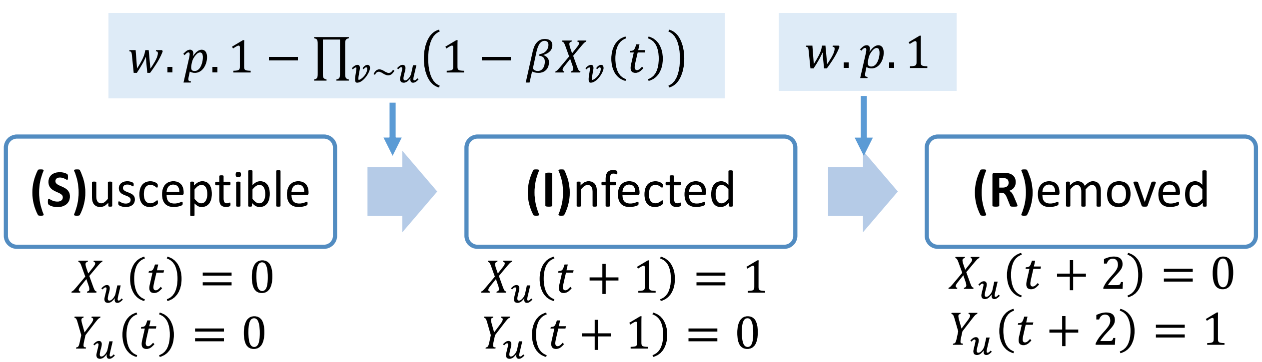
Outline

- Two *Classical* Epidemic Models
  - **S**: susceptible, **I**: infected, **R**: removed
  - SIR model (a.k.a. Reed-Frost model)
  - SIS model
- Analysis in *General* Graphs
  - Upper and lower bounds on *degree of diffusion* in terms of graph theoretic parameters and model parameters
  - SIR model: spread size
  - SIS model: extinction time
- Analysis in *Specific* Graphs
  - Complete graph
  - E-R graph, star-shaped graph, hypercube graph

SIR Model, a.k.a. Reed-Frost Model

- Undirected graph $G = (V, E)$
  - V: node set, E: edge set
  - n individuals, i.e., $n := |V|$

- Epidemic dynamics
  - Discrete-time model, i.e., $t \in \{0, 1, 2, \dots\}$
  - An infected node infects each of its susceptible neighbors w.p. β indepently



- $|X(t)| = \sum_{v \in V} X_v(t)$: number of infected nodes at time t
- $|Y(t)| = \sum_{v \in V} Y_v(t)$: number of removed nodes at time t

Analysis on SIR Model

- *Degree of diffusion*: spread size, i.e., $\mathbf{E}[|Y(\infty)|]$
- Notations
  - A: the adjacency matrix of G, i.e., $A_{uv} = 1$ if $(u, v) \in E$
  - ρ: the largest absolute eigenvalue of A, a.k.a. *spectral radius*
  - β: the infection probability
- Note on spectral radius
  - $\|\cdot\|$: *operator norm* for matrix, Euclidean norm for vector

$$\|A\| := \max_{\|x\|=1} \|Ax\|$$

$$= \rho \qquad \text{if } A \text{ is symmetric}$$

  - Large spectral radius means well-connected graph

Analysis on SIR Model

- Degree of diffusion: spread size, i.e., $\mathbf{E}[|Y(\infty)|]$
- Notations
  - A: the adjacency matrix of G, i.e., $A_{uv} = 1$ if $(u, v) \in E$
  - ρ: the largest absolute eigenvalue of A, a.k.a. *spectral radius*
  - β: the infection probability
- Result on general graph: An upper bound on the degree of diffusion

Theorem 8.1 *Suppose $\beta\rho < 1$, where ρ is the spectral radius of the adjacency matrix A. Then the total number $|Y(\infty)|$ of nodes removed satisfies*

$$\mathbf{E}\left[|Y(\infty)|\right] \leq \frac{1}{1-\beta\rho}\sqrt{n|X(0)|},$$

where $|X(0)|$ is the number of initial infectives.

If the graph G is regular (i.e. each node has the same number of neighbours) with node degree d, then

$$\mathbf{E}\left[|Y(\infty)|\right] \leq \frac{1}{1-\beta\rho}|X(0)| = \frac{1}{1-\beta d}|X(0)|.$$

Proof of Theorem 8.1 (1)

- Using the union bound for each path of infections with length t,

$$\boldsymbol{P}(X_v(t)=1) \le \sum_{u_0,\ldots,u_t:(u_{i-1},u_i)\in E, u_t=v} \beta^t X_{u_0}(0)$$

- Since the uv-th entry of A^t is the number of paths of length t,

$$\begin{aligned}\mathbf{E}[|Y(\infty)|] &= \sum_{v\in V} \boldsymbol{P}(Y_v(\infty)=1) \\ &\le \sum_{v\in V} \sum_{t=0}^{\infty} \sum_{u_0,\ldots,u_t:(u_{i-1},u_i)\in E, u_t=v} \beta^t X_{u_0}(0) \\ &= \sum_{t=0}^{\infty} \sum_{u\in V} (\beta^t A^t)_{uv} X_u(0) \\ &= \sum_{t=0}^{\infty} e^T (\beta A)^t X(0)\end{aligned}$$

$$// \quad e = (1,1,\ldots,1)^T$$

- Since $\|\beta A\| = \beta\rho < 1$ and A is symmetric, we can write

$$\sum_{t=0}^{\infty} (\beta A)^t = (I - \beta A)^{-1}$$

Proof of Theorem 8.1 (2)

- By the Cauchy-Swartz inequality and the definition of the operator norm,

$$\mathbf{E}[|Y(\infty)|] = \sum_{t=0}^{\infty} e^T(\beta A)^t X(0)$$

$$= e^T(I - \beta A)^{-1} X(0)$$

$$\leq \|e\| \|(I - \beta A)^{-1} X(0)\| \quad (\because \text{Chachy-Swartz ineq.})$$

$$\leq \|e\| \|(I - \beta A)^{-1}\| \|X(0)\| \quad (\because \text{ Def. of operator norm})$$

$$= \|(I - \beta A)^{-1}\| \sqrt{n|X(0)|}$$

- Noting the spectral radius of $(I - \beta A)^{-1}$ is $(1 - \beta\rho)^{-1}$ and $(I - \beta A)^{-1}$ is symmetric, we have

$$\|(I - \beta A)^{-1}\| = (1 - \beta\rho)^{-1}$$

Proof of Theorem 8.1 (3)

- (**Homework**) Complete the proof of the second part
- → Now suppose graph G is d-regular, i.e., $\sum_{v\in V} A_{uv} = d$.

$$\mathbf{E}[|Y(\infty)|] = \sum_{t=0}^{\infty} e^T(\beta A)^t X(0)$$
$$= e^T(I - \beta A)^{-1} X(0)$$
$$= \ldots$$

Theorem 8.1 *Suppose $\beta\rho < 1$, where ρ is the spectral radius of the adjacency matrix A. Then the total number $|Y(\infty)|$ of nodes removed satisfies*

$$\mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta\rho}\sqrt{n|X(0)|},$$

where $|X(0)|$ is the number of initial infectives.

If the graph G is regular (i.e. each node has the same number of neighbours) with node degree d, then

$$\mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta\rho}|X(0)| = \frac{1}{1-\beta d}|X(0)|.$$

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SIS Model, e.g., flu

- Undirected graph $G = (V, E)$
  - V: node set, E: edge set
  - n individuals, i.e., $n := |V|$

- Continuous-time Markov chain with state space $\{0, 1\}^n$
  - 0: susceptible, 1: infected
  - Non-zero transition rate $q(x, y)$

$$\textit{Birth rate: } q(x, x + e_i) = \beta(1 - x_i) \sum_{j \sim i} x_j$$
$$\textit{Death rate: } q(x, x - e_i) = x_i$$

  - A node can be infected and recover multiple times and the chain will absorb at all-0 state
    → Degree of diffusion = extinction time, i.e., time to all-0 state denoted by τ

Analysis on Fast Extinction of SIS Model (1)

- An upper bound on the extinction time τ

Theorem 8.2 *Let A denote the adjacency matrix of graph G, and ρ denote the spectral radius of this matrix. Then for any initial condition* $X(0) = \{X_i(0)\}_{i=1,\ldots,n}$, *and all* $t \geq 0$, *one has the following:*

$$\mathbf{P}(X(t) \neq 0) \leq \sqrt{n \sum_{i=1}^{n} X_i(0)} \exp\left((\beta\rho - 1)t\right) , \tag{8.4}$$

where $X(t) := \{X_i(t)\}_{1=1,\ldots,n}$ *denotes the state of the contact process with parameter* β, *on graph G, at time t.*

- Note that $\mathbf{P}(X(t) \neq 0) = \mathbf{P}(\tau \geq t)$

Analysis on Fast Extinction of SIS Model (2)

- An upper bound on the extinction time τ

Corollary 8.6 *Consider the contact process on a finite graph G on n nodes, with base infection rate β and arbitrary initial condition $X(0) \in \{0,1\}^n$. Let τ denote the time to absorption at 0 by the process. Then, under the condition*

$$\beta\rho < 1\,, \tag{8.10}$$

where ρ is the spectral radius of the adjacency matrix of G, it holds that

$$\boxed{\mathbf{E}(\tau) \le \frac{\log n + 1}{1 - \beta\rho}}. \quad \text{Fast extinction!} \tag{8.11}$$

Proof Write

$$\begin{aligned}
\mathbf{E}(\tau) &= \int_0^\infty \mathbf{P}(\tau > t)\,dt \\
&= \int_0^\infty \mathbf{P}(X(t) \neq 0)\,dt \\
&\le \int_0^\infty \min\left(1, n\exp(-(1-\beta\rho)t)\right) dt \\
&= t^* + \int_{t^*}^\infty n\exp(-(1-\beta\rho)t)\,dt,
\end{aligned}$$

where $t^* = (\log n)/(1-\beta\rho)$. We thus obtain

$$\mathbf{E}(\tau) \le t^* + \frac{n}{1-\beta\rho}\exp(-(1-\beta\rho)t^*) = \frac{\log n + 1}{1-\beta\rho}\,.$$

□

Analysis on Long Survival of SIS Model (1)

- An lower bound on the extinction time τ
- Notation
  - Isoperimetric constant $\eta(m)$ of graph G given m
  - Large $\eta(m)$ means that any k $(\leq m)$ nodes are highly influential

Definition 8.7 (Isoperimetric constant) For a graph G on the node set $\{1, \ldots, n\}$, and any integer $m < n$, the *isoperimetric constant* $\eta(m)$ of graph G is defined by

$$\eta(m) = \min_{S \subset \{1,\ldots,n\}, |S| \leq m} \frac{E(S, \bar{S})}{|S|}, \tag{8.12}$$

where $\bar{S}$ denotes the complementary set $\{1, \ldots, n\} \setminus S$, and $E(S, T)$ denotes the number of edges with one endpoint in set S and the other in set T.

Analysis on Long Survival of SIS Model (2)

- An lower bound on the extinction time τ

Theorem 8.8 *Let a finite graph G on n nodes be given, and assume that for some $m < n$ and some $r \in (0, 1)$, it holds that*

$$\beta\eta(m) \geq \frac{1}{r}, \tag{8.13}$$

where $\eta(m)$ denotes the isoperimetric constant of G. Then, denoting by τ the time to absorption of the contact process on G, for any initial condition $X(0) \neq 0$, it holds that:

$$\mathbf{P}\left(\tau \geq \frac{s}{2m}\right) \geq \frac{1-r}{1-r^m}\left(\frac{1-r^{m-1}}{1-r^m}\right)^s \left(1 - o(s^{-1})\right), \quad s \in \mathbb{N}, \tag{8.14}$$

where the term $o(s^{-1})$ is independent of the model parameters.

Analysis on Long Survival of SIS Model (3)

- An lower bound on the extinction time τ
- Notations
  - Isoperimetric constant $\eta(m)$ of graph G given m

Corollary 8.9 *Consider a sequence of finite graphs G_n on n nodes, a base infection rate β_n and an integer $m_n \geq n^a$, where a is a fixed positive constant, such that*

$$\beta_n \eta(m_n, G_n) \geq \frac{1}{r} , \tag{8.17}$$

where $r \in (0, 1)$ is fixed. Then, denoting by τ_n the time to extinction of the contact process on G_n, with parameter β_n, it holds that

$$\mathbf{E}(\tau_n) \geq \exp(bn^a) , \tag{8.18}$$

for some positive constant $b > 0$.

Proof) (**Homework**)

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- Analysis in *Specific* Graphs
  - Complete graph
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SIR Model in Complete Graph (1)

- Suppose one node was initially infected, i.e., $|X(0)| = 1$
- Analysis1. a sufficient condition for *small infection*
  - Noting complete graph is $(n-1)$-regular graph, i.e., $\rho = (n-1)$, from the second part of Theorem 8.1, it directly follows that

$$\text{If } \beta < \frac{1}{(n-1)}, \mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta(n-1)}$$

Theorem 8.1 *Suppose $\beta\rho < 1$, where ρ is the spectral radius of the adjacency matrix A. Then the total number $|Y(\infty)|$ of nodes removed satisfies*

$$\mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta\rho}\sqrt{n|X(0)|},$$

where $|X(0)|$ is the number of initial infectives.

If the graph G is regular (i.e. each node has the same number of neighbours) with node degree d, then

$$\mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta\rho}|X(0)| = \frac{1}{1-\beta d}|X(0)|.$$

SIR Model in Complete Graph (2)

- Suppose one node was initially infected, i.e., $|X(0)| = 1$
- Analysis1. a sufficient condition for *small infection*
  - Noting complete graph is $(n-1)$-regular graph, i.e., $\rho = (n-1)$, from the second part of Theorem 8.1, it directly follows that

$$\text{If } \beta < \frac{1}{(n-1)}, \mathbf{E}[|Y(\infty)|] \leq \frac{1}{1-\beta(n-1)}$$

- Analysis2. a sufficient condition for *large infection*
  - Recalling Theorem 2.1 (ii) (the giant component of E-R graph in supercritical regime),

Theorem 8.11 *Let γ be the unique positive solution of $\gamma + e^{-\gamma c} = 1$. Then, as $n \to \infty$, the size of the largest connected component in the random graph $G(n, \beta)$ is $(1 + o(1))\gamma n$, with probability going to 1 as n tends to infinity.*

Theorem 8.12 *Let $G = (V, E)$ be the complete graph on n nodes, and let $\beta = \frac{c}{n-1}$ for an arbitrary constant $c > 1$. Then, the final size of the epidemic satisfies*

$$\mathbf{E}[|Y(\infty)|] \geq (1 + o(1))\gamma^2 n,$$

for any $|X(0)| \geq 1$, where $\gamma > 0$ solves $\gamma + e^{-\gamma c} = 1$. Moreover, $|Y(\infty)| = O(\log n)$ with probability $1 - \gamma$.

SIS Model in Complete Graph

- Suppose one node was initially infected, i.e., $|X(0)| = 1$
- Analysis1. a sufficient condition for *fast extinction*
  - Note that $\rho = (n-1)$
  - From Corollary 8.6, it follows that

$$\text{If } \beta < \frac{1}{(n-1)}, \mathbf{E}(\tau) \leq \frac{\log n + 1}{1 - \beta(n-1)}$$

- Analysis2. a sufficient condition for *long survival*
  - Check that $\eta(m) = n - m$
  - From Corollary 8.9, it follows that for given constant $a > 0$, there exists a constant $b > 0$ such that

$$\text{If } \beta > \frac{1}{(n - n^a)}, \mathbf{E}(\tau) \geq \exp(bn^a)$$

Preliminary to Proof of Theorems 8.2&8.8

- Skip-free Markov jump process with state space $\mathbb{N}^K$
  - Non-zero transition rate $q(x, y)$

$$\textit{Birth rate: } q(x, x + e_i) = \beta_i(x)$$
$$\textit{Death rate: } q(x, x - e_i) = \delta_i(x)$$

- SIS model is a Markov jump process with the following birth and death rates

State space $\{0,1\}^n$

$$q(x, x + e_i) = \beta(1 - x_i) \sum_{j \sim i} x_j$$
$$q(x, x - e_i) = x_i$$

⟷

State space $\{0,1,2, \dots\}^n$

$$\beta_i^c(x) = \mathbf{1}_{x_i=0} \beta \sum_{j \sim i} x_j$$
$$\delta_i^c(x) = x_i$$

- Sketch of the proofs
  - Consider an analytically tractable process $X^{brw}(t)$ or $Z(t)$
  - Construct the coupling between the original process and the tractable process which provides a stochastic dominance
  - Combine the analysis on the tractable process and the stochastic dominance

Proof of Theorem 8.2 (1)

- Branching random walk process $X^{brw}(t)$ on $\mathbb{N}^n$
  - A skip-free Markov jump process with birth rate $\beta_i^{brw}(x)$ and death rate $\delta_i^{brw}(x)$

Branching random walk $X^{brw}(t)$

State space $\{0,1,2,\dots\}^n$

$$\beta_i^{brw}(x) = \beta \sum_{j\sim i} x_j$$
$$\delta_i^{brw}(x) = x_i$$

SIS model $X^c(t)$

State space $\{0,1,2,\dots\}^n$

$$\beta_i^c(x) = \mathbf{1}_{x_i=0}\beta \sum_{j\sim i} x_j$$
$$\delta_i^c(x) = x_i$$

- Comparison of $X^c(t)$ with $X^{brw}(t)$
  - Lower birth rate and higher death rate

$$\beta_i^{brw}(x) = \beta \sum_{j\sim i} x_j \;\geq\; \beta_i^c(x) = \mathbf{1}_{x_i=0}\beta \sum_{j\sim i} x_j$$
$$\delta_i^{brw}(x) = x_i \;\leq\; \delta_i^c(x) = x_i$$

→ $\left|X^{brw}(t)\right| \geq_{st} |X^c(t)|$

  - Proof using the coupling technique

Stochastic Dominance from Coupling

Theorem 8.4 *Consider two skip-free Markov jump processes X, X' defined on the state space $\mathbb{N}^K$, with respective birth rates $\beta_i(x)$, $\beta_i'(x)$ and death rates $\delta_i(x)$, $\delta_i'(x)$, for $x \in \mathbb{N}^K$ and $i \in \{1, \ldots, K\}$.*

Assume that for all $x, y \in \mathbb{N}^K$ such that $x \le y$ (i.e. $x_i \le y_i$ for all $i = \{1, \ldots, K\}$), the following holds:

$$x_i = y_i \Rightarrow \beta_i(x) \le \beta_i'(y) \text{ and } \delta_i(x) \ge \delta_i'(y). \tag{8.5}$$

Then, for initial conditions $X(0)$ and $X'(0)$ satisfying $X(0) \le X'(0)$, one can construct the two processes X, X' jointly so that for all $t \ge 0$, the ordering is preserved, that is $X(t) \le X'(t)$.

→ i.e., $|X(t)| \le_{st} |X'(t)|$

- Proof by coupling $X(t)$ (or $X^c(t)$) and $X'(t)$ (or $X^{brw}(t)$) as follows:

If $x_i < x_i'$

$$\begin{aligned}
q((x,x'),(x+e_i,x')) &= \beta_i(x)\,,\\
q((x,x'),(x,x'+e_i)) &= \beta_i'(x')\,,\\
q((x,x'),(x-e_i,x')) &= \delta_i(x)\,,\\
q((x,x'),(x,x'-e_i)) &= \delta_i'(x')\,.
\end{aligned}$$

If $x_i = x_i'$

$$\begin{aligned}
q((x,x'),(x+e_i,x'+e_i)) &= \beta_i(x)\,,\\
q((x,x'),(x,x'+e_i)) &= \beta_i'(x')-\beta_i(x)\,,\\
q((x,x'),(x-e_i,x'-e_i)) &= \delta_i'(x')\,,\\
q((x,x'),(x-e_i,x')) &= \delta_i(x)-\delta_i'(x')\,.
\end{aligned}$$

Marginal transition rate of $X(t)$

$$q(x,x+e_i) = \textstyle\sum_{x',y'} q\big((x,x'),(x+e_i,y')\big) = \beta_i(x)$$

$$q(x,x-e_i) = \textstyle\sum_{x',y'} q\big((x,x'),(x-e_i,y')\big) = \delta_i(x)$$

Marginal transition rate of $X'(t)$

$$q(x',x'+e_i) = \beta_i'(x)$$

$$q(x',x'-e_i) = \delta_i'(x)$$

A rigorous proof is provided in p.92-p.94 (**Homework**)

Proof of Theorem 8.2 (2)

- The coupling construction with $X^{brw}(0) = X^c(0) = X(0)$ implies $X^{brw}(t) \geq_{st} X^c(t)$, i.e.,

$$\begin{aligned}\mathbf{P}(X^c(t) \neq 0) &\leq \mathbf{P}\left(X^{brw}(t) \neq 0\right) \\ &\leq e^T \mathbf{E}\left(X^{brw}(t)\right)\end{aligned}$$

- From the linear structure of the transition rates of the branching random walk,

$$\frac{d}{dt}\mathbf{E}\left(X^{\text{brw}}(t)\right) = \beta A\,\mathbf{E}\left(X^{\text{brw}}(t)\right) - \mathbf{E}\left(X^{\text{brw}}(t)\right) \quad \text{thus} \quad \mathbf{E}\left(X^{\text{brw}}(t)\right) = \exp\left(t(\beta A - I)\right) X(0)$$

  - where $\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$
  - See https://en.wikipedia.org/wiki/Matrix_differential_equation for details

- By the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbf{P}(X^c(t) \neq 0) &\leq e^T \mathbf{E}\left(X^{brw}(t)\right) = e^T \exp\left(t(\beta A - I)\right) X(0) \\ &\leq \|e\| \left\| \exp\left(t(\beta A - I)\right) X(0) \right\|\end{aligned}$$

Proof of Theorem 8.2 (3)

- Recalling the operator norm of a symmetric matrix is its spectral radius,

$$\begin{aligned}\mathbf{P}(X^c(t) \neq 0) &\le \|e\| \left\| \exp\big(t(\beta A - I)\big) X(0) \right\| \\ &\le \|e\| \left| \exp\big(t(\beta\rho - 1)\big) \right| \|X(0)\| \\ &= \sqrt{n \sum_{i=1}^{n} X_i^2(0)} \exp\big((\beta\rho - 1)t\big) \\ &= \sqrt{n \sum_{i=1}^{n} X_i(0)} \exp\big((\beta\rho - 1)t\big)\end{aligned}$$

- Summary of the proof
  - Consider the analytically tractable process $X^{brw}(t)$
  - Construct the coupling between $X^{brw}(t)$ and $X^c(t)$ providing the stochastic dominance
  - Combine the analysis on $X^{brw}(t)$ and the stochastic dominance to provide an *upper bound* of $\mathbf{P}(\tau \ge t)$

  - Note) The proof of Theorem 8.8 will be similar to this proof but we are interested in a *lower bound* of $\mathbf{P}(\tau \ge t)$ in Theorem 8.8

Proof of Theorem 8.8 (1)

- Consider the Markov jump process $Z(t)$ on $\{0,1,\dots,m\}$, with non zero transition rates as follows:

$$q(z,z+1) = r^{-1}\, z\, \mathbf{1}_{z<m}$$
$$q(z,z-1) = z$$

- Construct the joint process (or coupling) (X,Z) on $\{(x,z) \in \{0,1\}^n \times \{0,\dots,m\}, z \le \sum_{i=1}^n x_i\}$ with non-zero rates in the following:

If $\sum_{i=1}^n x_i > z$

$$\begin{aligned} q((x,z),(x+e_i,z)) &= \beta(1-x_i)\textstyle\sum_{j\sim i} x_j\,, \\ q((x,z),(x-e_i,z)) &= x_i\,, \\ q((x,z),(x,z+1)) &= r^{-1} z \mathbf{1}_{z<m}\,, \\ q((x,z),(x,z-1)) &= z\,. \end{aligned}$$

If $\sum_{i=1}^n x_i = z$

$$\begin{aligned} q((x,z),(x+e_i,z+1)) &= c_i(x)\,, \\ q((x,z),(x+e_i,z)) &= \beta(1-x_i)\textstyle\sum_{j\sim i} x_j - c_i(x)\,, \\ q((x,z),(x-e_i,z-1)) &= x_i\,, \end{aligned}$$

Marginal transition rate of $X(t)$

$$q(x,x+e_i) = \textstyle\sum_{x',y'} q\big((x,x'),(x+e_i,y')\big)$$
$$= \beta(1-x_i)\textstyle\sum_{j\sim i} x_j$$
$$q(x,x-e_i) = x_i$$

Marginal transition rate of $Z(t)$

$$q(z,z+1) = \begin{cases} r^{-1}\, z\, \mathbf{1}_{z<m} & \text{if } \sum_{i=1}^n x_i > z \\ \sum_{i=1}^n c_i(x) & \text{if } \sum_{i=1}^n x_i = z \end{cases}$$
$$q(z,z-1) = z$$

Proof of Theorem 8.8 (2)

- We need to show the existence of $c_i(x)$ such that
  - i) Transition rate is non-negative

$$0 \le c_i(x) \le \beta(1 - x_i) \sum_{j \sim i} x_j, \; i \in \{1, \dots, n\},$$

  - ii) Marginal transition rate of $Z(t)$

$$\sum_{i=1}^{n} c_i(x) = r^{-1} z \mathbf{1}_{z<m}$$

- To show the existence of $c_i(x)$, it is enough to show

$$\sum_{i=1}^{n} \beta(1 - x_i) \sum_{j \sim i} x_j \ge r^{-1} z \mathbf{1}_{z<m} .$$

  - Let S denote the set of nodes $j \in \{1, \dots, n\}$ such that $x_j = 1$
  - Then, $\sum_{i=1}^{n} \beta(1 - x_i) \sum_{j \sim i} x_j = \beta E\,(S, \bar{S})$
  - Note that $|S| = \sum_j x_j = z \le m$
  - Hence, by the definition of isoperimetric constant and the assumption on r,

$$\beta \frac{E(S, \bar{S})}{|S|} \ge \beta \eta(m) \ge r^{-1}$$

Proof of Theorem 8.8 (3)

- From the construction of coupling (X, Z), it follows that

$$P(\tau > s) \geq P(Z(s) = 0)$$

- To evaluate the right-hand side, consider the discrete-time Markov chain $Y(k)$ keeping track of the states visited by process $Z(t)$

$$\begin{aligned}
\mathbf{P}(Y(k+1) = y+1 \mid Y(k) = y) &= \frac{y/r}{y/r+y} = \frac{1}{1+r}, \; y \in \{1, \ldots, m-1\}, \\
\mathbf{P}(Y(k+1) = y-1 \mid Y(k) = y) &= \frac{y}{y/r+y} = \frac{r}{r+1}, \; y \in \{1, \ldots, m-1\}, \\
\mathbf{P}(Y(k+1) = m-1 \mid Y(k) = m) &= 1, \\
\mathbf{P}(Y(k+1) = 0 \mid Y(k) = 0) &= 1.
\end{aligned}$$

- Let $\pi_{k'}$ denote the probability that starting from state $k' \in \{0, \ldots, m\}$, the chain $Y(k)$ hits m before it is absorbed at 0

$$\pi_0 = 0, \; \pi_m = 1, \; (1+r)\pi_k = r\pi_{k+1} + \pi_{k-1}, \; k \in \{1, \ldots, m-1\},$$

(Homework) → $\pi_k = \frac{1-r^k}{1-r^m}$

Proof of Theorem 8.8 (4)

- Then, we have

$$P(\{Y(k)\}_{k\geq 0} \text{ visits state } m \text{ at least } s \text{ times}) \geq \frac{1-r}{1-r^m}\left(\frac{1-r^{m-1}}{1-r^m}\right)^s$$

- Also, after each entrance into state m, process $Z(t)$ remains there for an exponentially distributed random time, with mean $\frac{1}{m}$. Thus, if follows that

$$\mathbf{P}(Z(s/2m) > 0) \geq \mathbf{P}\left(\sum_{i=1}^{s} E_i \geq s/2\right)\frac{1-r}{1-r^m}\left(\frac{1-r^{m-1}}{1-r^m}\right)^s$$

  - where the random variables E_i are i.i.d., exponentially distributed with mean 1

- From Chernoff's Lemma 1.8,

$$\mathbf{P}\left(\sum_{i=1}^{s} E_i \geq s/2\right) \geq 1 - \exp\left(-s h_{\exp}(1/2)\right),$$

where

$$\begin{aligned} h_{\exp}(x) &= \sup_{\theta\in\mathbb{R}}\left(\theta x - \log \mathbf{E}(\exp(\theta E_1))\right) \\ &= \sup_{\theta\in\mathbb{R}}\left(\theta x - \log(1/(1-\theta))\right) \\ &= x - 1 - \log x . \end{aligned}$$

- The term $\exp\left(-s h_{exp}\left(\frac{1}{2}\right)\right)$ is clearly $o(s^{-1})$

Homework

- Prove the second part of Theorem 8.1
- Prove Corollary 8.9
- Prove Theorem 8.4
- Show $\pi_0 = 0,\ \pi_m = 1,\ (1 + r)\pi_k = r\pi_{k+1} + \pi_{k-1},\ k \in \{1, \ldots, m - 1\},$ → $\pi_k = \frac{1-r^k}{1-r^m}$