Coupling

우지인

JiinWoo@lanada.kaist.ac.kr

Contents

- Coupling method
- Stochastic dominance
- Paper 1: Stochastic dominance of Genie policy
- Paper 2: Stochastic dominance of LCQ

Coupling Method

- Proof technique that allows to compare two unrelated random variables(distributions) X and Y by creating random vector W = (X',Y') whose marginal distributions correspond to X and Y respectively.
- Make coupling so that X and Y can be related in a certain way we desire.

Definition 4.1 (Coupling). *Let μ and ν be probability measures on the same measurable space $(S, \mathcal{S})$. A* coupling *of μ and ν is a probability measure γ on the product space $(S \times S, \mathcal{S} \times \mathcal{S})$ such that the* marginals *of γ coincide with μ and ν, i.e.,*

$$\gamma(A \times S) = \mu(A) \quad \text{and} \quad \gamma(S \times A) = \nu(A), \qquad \forall A \in \mathcal{S}.$$

Coupling Method

- Typical Couplings
  - Independent Coupling
    - Coupling (X',Y') s.t. X'= X, Y'=Y are independent.
  - Maximal Coupling
    - Coupling (X',Y') s.t. $\|\mu - \nu\|_{\mathrm{TV}} = \inf\{\mathbb{P}[X \neq Y] : \textit{coupling } (X, Y) \textit{ of } \mu \textit{ and } \nu\}$
    - Used to show mixing time bound.
  - Monotone Coupling
    - Coupling (X',Y') s.t. P(X'>Y') = 1
    - Used to show stochastic dominance

Stochastic Dominance

- Quantifies the concept of one random variable being "bigger" than another.
- Stochastic Dominance (Usual stochastic order)

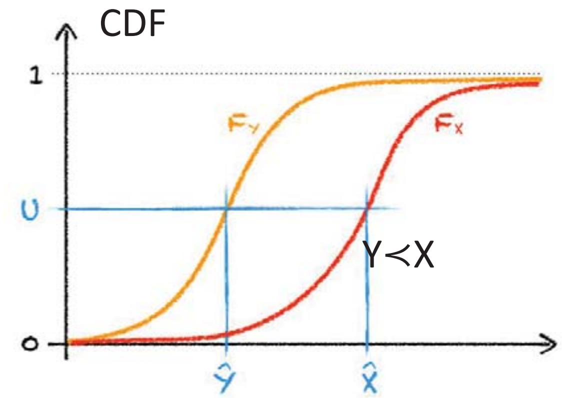
We say that the random variable X *first order stochastically dominates (FSD)* the random variable Y, written $X \succsim_{FSD} Y$, if

$$\Pr\{X > z\} \geq \Pr\{Y > z\} \quad \text{for all } z, \tag{3.1}$$

Theorem 4.23 (Coupling and stochastic domination). *The real random variable X stochastically dominates Y if and only if there is a coupling $(\hat{X}, \hat{Y})$ of X and Y such that*

$$\mathbb{P}[\hat{X} \geq \hat{Y}] = 1. \tag{4.1}$$

We refer to $(\hat{X}, \hat{Y})$ as a monotone coupling *of X and Y.*



Stochastic Dominance – Monotone coupling

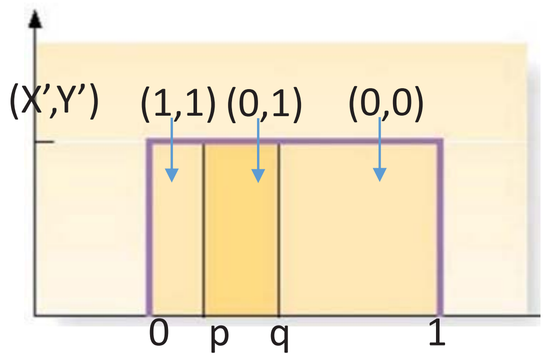
- Bernoulli variables X,Y s.t. P(X)=p, P(Y)=q (q>p)

  Pick U uniformly at random in [0, 1] and set X' = $1_{\{U<p\}}$, Y' = $1_{\{U<q\}}$
- Example (Compare binomial r.v.)

  Objective: $\sum X_i \leq_{st} \sum Y_i$:Total number of heads

  Assume two biased coins X, Y (1: head , 0: tail). P(X) = p , P(Y) =q (q>p)

  - if Xi = 1, then Yi = 1,
  - if Xi = 0, then Yi = 1 with probability (q-p)/(1-p).



Stochastic Dominance

- Use to compare random variable.
- Why compare random variables?
  - Some random variables can be metric of performance
  - Ex. Compare performance of process under policy.

- 2 papers compare performance of policy using coupling.
  - Connectivity Serving Content with Unknown Demand : the High-Dimensional Regime (SIGMETRICS-2014)
  - Dynamic Server Allocation to Parallel Queues with Randomly Varying (TIT-1993)