Lecture 10: (chapter 5) From microscopic to macroscopic

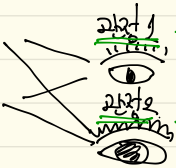
↳ "fluid model", "~~mea~~ mean-field approximation"

"Fluid models for Internet"

(System (Dynamical system))
model

system

자전거, social network, TCP/IP,



관점 1 → 결론 1.

관점 2 → 결론 2.

보이는 것이 달라진다. → 판단, knowledge 달라진다.

$\{X_i\}$, i.i.d

$$\frac{\sum_{i=1}^{n} X_i}{n}$$

관점 1: 개개 R.V.의 sum의 평균: stochastic 관점 세상

관점 2: n을 키워보자 → 숫자 1개로 보인다: deterministic 한 세상.

대부분이 (MC)

Microscopic: system 안의 (변화/움직임)을 한 step/한 step modeling을 해서 본다.
상태변화

분석 / tractability

Macroscopic: 개개의 상태변화에서 벗어나서 "큰그림"에서 system이 어떻게 행동

"파악한다" approximation → 결론을 내릴 수 있는 것들이 있다. → 조심하려면

(Q) 보자. Microscopic한 model 생각하는 → Macroscopic model을 원한 (?)

$\dot{x}(t) = f(x(t))$ 1. what 2. 수학적 연결

"변형" → 수학적으로 뒷받침이 되는

Example
① Poisson process
② M/M/1 Queue
③ SIS epidemics

Micro
Macro

③

$$\Pr\left(X_m(t+h) = i+1 \mid X_m(t) = i\right) = h\frac{\beta}{m} i(m-i) + o(h)$$

$$\Pr\left(X_m(t+h) = i-1 \mid X_m(t) = i\right) = \delta i h + o(h)$$

$$\Pr\left(X_m(t+h) = i \mid X_m(t) = i\right) = 1 - h\left(\frac{\beta}{m} i(m-i) + \delta i\right) + o(h)$$

microscopic

맞나?

$X_m(t)$: t시간에 감염된 숫자

$x(t)$: 전체에서 감염된 것의 비율

무한개 → 수학적으로 show

$$\boxed{\dot{x}(t) = \beta x(1-x) - \delta x}$$

Macroscopic

fluid model   mean-field approximation

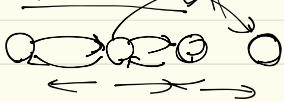
그렇긴하지만, 맞는지는 모르겠다.

평균 → approximation

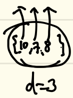
Kurtz Theorem ('chapter 5.3) :

Markov jump process (Poisson process 를 일반화,
Birth-Death 를 일반화)

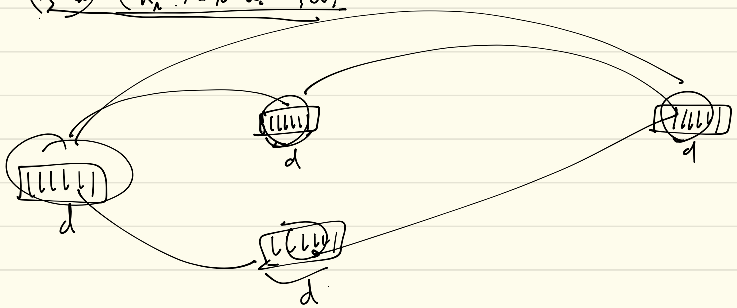
M/M/1 Queue



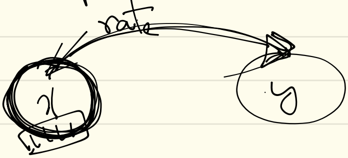
- Population of individuals belonging to "d" difference species



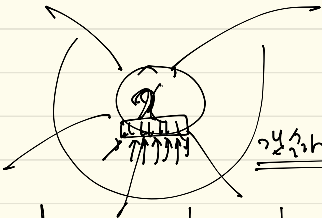
- $X_i(t)$, $i = 1, \dots, d$: total # of individuals of species i at time t
- $X(t) = (X_i(t) : i = 1, \dots, d)$:



- We say that $X(t)$ is $\boxed{\text{Markov jump process}}$ when for any given two states $x, y \in \mathbb{N}^d$, the process jumps from x to y upon expiration of a random timer with exponential distribution whose parameter depends on x and y.



- Assume that $\exists$ a finite # of possible jump directions (e_i) $i = 1, \ldots, k$



갯수가 key

(질문) k와 d의 관계는? 관계는 없다.

(많은경우 k=d가 되긴하지) $k < d$

possible # of jump direction $\leq$ ~~d~~ 2^d

(cf) When e_i's entries are restricted to $\{-1, 0, 1\}$ $\Rightarrow$ Birth-and-death process

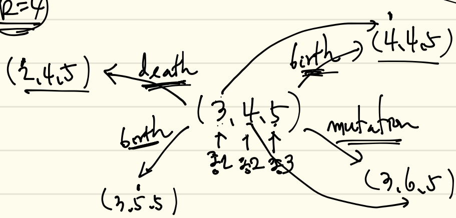
(예) M/M/1

(cf) Poisson process $e_i \to$ ~~...~~ 0 또는 1 $e_i \in \mathbb{N}^d$

Let us denote by $\lambda_i(x)$ " the rate from x to $x + e_i$

$\to$ exponential time 동안 머무르다가 $\to$ 평균 $\frac{1}{\lambda_i(x)}$

(EX) $d=3$, $k=4$



(아이디어): scaling parameter = n (나중에 $n \to \infty$)

For any ~~given~~ fixed n $\longrightarrow$ MJP(n), $X_n(t)$

$n=1 \longrightarrow X_1(t)$

$2 \longrightarrow X_2(t)$

$\vdots$

$n \longrightarrow X_n(t)$

$\vdots$

$\infty \longrightarrow X_\infty(t)$ Question $X_\infty(t)$가 (어떻게 행동할것인가? / 아름답게 행동 $\leftarrow$ 조건)

$\boxed{\text{Thm 5.5의 가정}} \longrightarrow \boxed{\text{Thm 5.5}}$

Thm 5.5 (Kurtz's Theorem) (page 67의 아래부분)

$n \longrightarrow$ MJP(n) / $(X_n(t))_{t \in \mathbb{R}}$ on $\mathbb{N}^d$ with jump directions $(e_i)_{i=1,\dots,k}$

transition rates $n\lambda_i(x/n)$, $x \in \mathbb{N}^d$ (이렇게 scaling이 되었다면)

$x \longrightarrow x + e_i$.

Then, $\{X_n(t)\}_{t \geq 0}$ satisfies:

$$\begin{pmatrix} \Pr(X_n(t+h) = x + e_i \mid X_n(t) = x) = n \cdot h\, \lambda_i(x/n) + o(h), \quad x \in \mathbb{N}^d,\ i = 1, \dots, k \\ \Pr(X_n(t+h) = x \mid X_n(t) = x) = 1 - nh \sum_{j=1}^{k} \lambda_i(x/n) + o(h), \quad x \in \mathbb{N}^d \end{pmatrix}$$

- Let the function $F : \mathbb{R}^d \to \mathbb{R}^d$ be defined by, $e_1, \dots, e_k$

$$\Longrightarrow F(x) := \sum_{i=1}^{k} e_i \lambda_i(x) = \underbrace{(\circ, \circ, \circ, \dots, \circ)}_{d\ \text{dim}}$$

- Let $\bar{e} := \max_{i=1\dots k} |e_i|$. $\to$ some arbitrary (norm / distance)

Assume the following conditions hold,

C1: $\bar{\lambda} := \max_{i=1,\dots,k} \sup_{x\in\mathbb{R}^d} \lambda_i(x)$ is finite. ← (대충 만족하겠지....)

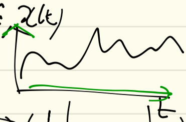
C2: The function $\bar{F}$ is ~~Lip~~ Lipschitz continuous, i.e, $\exists$ a constant M s.t

$$|\bar{F}(x)-\bar{F}(y)| \le M|x-y| \quad \text{for all } x,y\in\mathbb{R}^d$$

$$\frac{|\bar{F}(x)-\bar{F}(y)|}{|x-y|} \le M$$

C3: $\lim_{n\to\infty} \frac{1}{n}X_n(0) = x(0)$ a.s (대충 만족합니다).

Let $x: \mathbb{R}_+ \to \mathbb{R}^d$ be the solution of the following integral equation: $x(t)$



시간에 대한 함수

Macro 세상은 B응

$$\boxed{x(t) = x(0) + \int_0^t \bar{F}(x(s))\,ds}$$

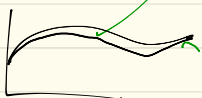
(deterministic)

Then, we have: for any fixed $\varepsilon, T > 0$, for sufficiently large n

$$P_r\left(\sup_{0\le t\le T}\left|\frac{X_n(t)}{n} - x(t)\right| \ge \varepsilon\right) \le 2k\exp\left(-nT\bar{\lambda}\,h\left(\frac{\varepsilon e^{-MT}}{2kT\bar{\lambda}e}\right)\right),$$

where $h(t) = (1+t)\log(1+t) - t$. Moreover

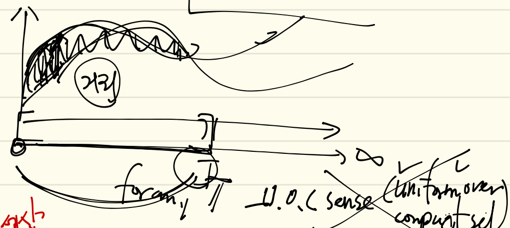
$$\boxed{\lim_{n\to\infty} \sup_{0\le t\le T}\left|\frac{1}{n}X(t) - x(t)\right| = 0} \quad \text{a.s.}$$



$\boxed{\frac{X_n(t)}{n} \xleftrightarrow{\text{거의 같다}} x(t)}$ (for large n)

시간에 대한 함수

두 함수가 거의 비슷 (거리를 define)



확률적 세상 → 결정론적 세상

Kurtz theorem: 대수의 법칙 → 시간 t에 대한 functional version

"functional law of large numbers"

central limit theorem

Lecture 11 (part 2) : Kurtz Theorem의 응용

(① Poisson process
② SIS epidemics)

original system $\lambda_i(x)$, $X(t)$ $\longrightarrow$ n-scaled system $n\cdot\lambda_i\left(\frac{x}{n}\right)$, $X_n(t)$

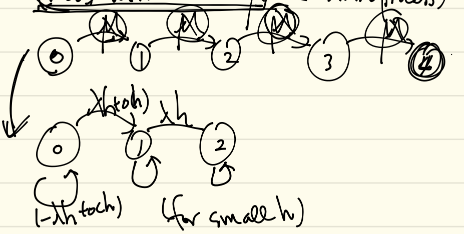
$\downarrow$ F

$\downarrow$ 미분방정식 $\rightarrow x(t)$

$\frac{X_n(t)}{n} \approx x(t)$

<Poisson process>

$N(t)$ with rate μ (Birth process)



$N(t) \approx \boxed{}$

deterministic한 system, 함수

$N(t) \xrightarrow{\text{만들었다}}$ 이것의 n-scaled system $N_n(t)$

$\lambda(i) = \mu$ $\qquad \lambda_i = n\cdot\lambda\left(i/n\right) = n\cdot\mu$

By Kurtz theorem, $\frac{N_n(t)}{n} \sim \boxed{\mu t}$ $\qquad (N(t))$

$\dot{x}(t) = F(x(t))$, $F = \mu$

$= \mu \rightarrow x(t) = \mu t$

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| N(t) - \mu t \right| = 0 \quad \text{a.s} \qquad (N(t) \approx \mu t)$$

Proposition 5-2 $\qquad$ (증명X)

$N(nt) \rightarrow$ 시간을 빠르게

$N(t) \approx \frac{N_n(t)}{n} = \frac{N(nt)}{n}$

time에 관해서 빠르게

space에 관해서 더 작게

$N(t)$에 비해서

$x(2t)$ $\quad x(t)$

"Functional law of large #s"

$\frac{N(nt)}{n} \sim \frac{n\cdot N(t)}{n}$ $\qquad N(nt) = \cancel{N(t)}\ n\cdot N(t)$

$\frac{N(nt)}{n} \rightarrow \mu t$

$E(N(t)) = \mu t$

<SIS epidemics>

$$\begin{cases} P_r\left(X_n(t+h) = i+1 \mid X_n(t) = i\right) = h\left(\frac{\beta}{n} i(n-i)\right) + o(h) \\ P_r\left(X_n(t+h) = i-1 \mid X_n(t) = i\right) = \delta i h + o(h) \\ P_r\left(X_n(t+h) = i \mid X_n(t) = i\right) = 1 - h\left(\frac{\beta}{n} i(n-i) + \delta i\right) + o(h) \end{cases}$$

모델 setup에 "n"이라는 scaling parameter 존재

n명짜리 system

$$i-1 \xleftarrow{\delta i h} i \xrightarrow{\frac{\beta}{n} i(n-i)} i+1$$

$i \to i-1 : \delta i h = n \cdot \lambda_n^-(i) = n \cdot \lambda^-(i/n)$

$i \to i+1 : \frac{\beta}{n} i(n-i) = n \cdot \lambda^+(i/n)$

what is $\lambda^-(i)$? $\quad n \cdot \lambda^-\left(\frac{i}{n}\right) = \delta i h = n \cdot \delta\left(\frac{i}{n}\right) \cdot h \Longrightarrow \lambda^-(i) = \delta i$

$\lambda^+(i)$? $\quad n \cdot \lambda^+\left(\frac{i}{n}\right) = \frac{\beta}{n} i(n-i) = n \cdot \beta\left(\frac{i}{n}\right)\left(1 - \frac{i}{n}\right) \Rightarrow \lambda^+(i) = \beta i(1-i)$

$\bar{h} \to x$

$e_+ = 1, \quad e_- = -1$

$\bar{h}(i) = e_+ \lambda^+(i) + e_- \lambda^-(i) = \beta i(1-i) - \delta i$

$\bar{h}(x) = \beta x(1-x) - \delta x$ $\qquad$ For simplicity, $\delta = 1$, $x(0) = y$

$$\boxed{\dot{x}(t) = \bar{h}(x(t)) = \beta x(1-x) - \delta x} \longrightarrow \boxed{\text{homework}}$$

$$\boxed{x(t) = \frac{(\beta-1) y e^{(\beta-1)t}}{(\beta-1) - \beta y\left(1 - e^{(\beta-1)t}\right)}}$$

$\frac{X_n(t)}{n} \sim x(t)$

→ n이 충분히 클 때 감염된 node들의 "portion" "fraction"

① 미분방정식

② 직접 풀어볼 것

If $\beta > 1$, $\quad x(t) \to \left(1 - \frac{1}{\beta}\right)$ as $t \to \infty$

If $\beta < 1$, $\quad x(t) \to 0$ as $t \to \infty$

Lecture 11 (part 3) Kurtz Theorem 의 증명

$\boxed{\text{Proposition (5.2)}}$ (Kurtz Theorem 의 poisson process version) (쉬운 version)

Let $N(t)$ be a poisson process with unit rate ($\lambda = 1$). Then for any $\varepsilon > 0$ and $T > 0$,

$$\Pr\Big(\sup_{0 \le t \le T} |N(t) - t| \ge \varepsilon\Big) \le 2 e^{-T h(\varepsilon / T)}, \quad h(t) = (1+t)\log(t+1) - t.$$

(Pf) Doob's inequality: $N(t) - t$, $t - N(t)$ 이 만약 submartingale 이라면,

$$\Pr\Big(\sup_{0 \le t \le T} |N(t) - t| \ge \varepsilon\Big)$$

$$\le \underbrace{\Pr\Big(\sup_{0 \le t \le T} N(t) - t \ge \varepsilon\Big)}_{①} + \underbrace{\Pr\Big(\sup_{0 \le t \le T} t - N(t) \ge \varepsilon\Big)}_{②}$$

for any $\theta > 0$

① $$\Pr\Big(\sup_{0 \le t \le T} (N(t) - t) \ge \varepsilon\Big) = \Pr\Big(\sup_{0 \le t \le T} e^{(N(t)-t)\theta} \ge e^{\theta\varepsilon}\Big) \le \frac{E\left(e^{(N(T)-T)\theta}\right)}{e^{\theta\varepsilon}}$$

(Q) ⇒ 왜 $N(t) - t$ 가 submartingale 이냐?

(Continuous version of Martingale definition

$\{X(t)\}_{t \ge 0}$ is a submartingale if and only if the following holds:

$$E\big(X(t) \mid \{X(u)\}_{u \le s}\big) \ge X(s), \quad 0 \le s \le t$$ 모든)

$$E\big(N(t) - t \mid \{N(u) - u\}_{u \le s}\big) = E\big(N(t-s) + N(s) - (t-s) - s \mid \underbrace{\{N(u) - u\}_{u \le s}}_{D}\big)$$

왜? ("independent increments")

$$= E\big(N(t-s) - (t-s) \mid D\big) + E\big(N(s) - s \mid D\big) = N(s) - s$$

$$= \underbrace{E(N(t-s))}_{t-s} - (t-s)$$

s까지 알려졌으니 $N(s) - s$

$N(T) \sim$ Poisson random variable with parameter T

T고정

$$\frac{E\left(e^{(N(T)-T)\theta}\right)}{e^{\theta\varepsilon}} = e^{-\theta\varepsilon}\cdot e^{-T\theta}\, E\left(e^{N(T)\theta}\right)$$

$$= e^{-\theta\varepsilon}\cdot e^{-T\theta}\cdot e^{T(e^{\theta}-1)} = e^{\left(-\theta(\varepsilon+T)+T(e^{\theta}-1)\right)}$$

$$\theta^* = \log(1+\varepsilon/T)$$

when $\theta=\theta^*$

$$\doteq e^{-T h(\varepsilon/T)}, \quad h(t) = \text{-----}$$

해보시오 (숙제 (x))

(Proof) $\boxed{\frac{1}{n}X_n(t) \xleftrightarrow{\text{가까워진다}} x(t)}$ ← 미분방정식의 solution

$X_n(t)$: n번째의 system

① $X_n(t)$의 dynamics를 정의본다. $X_n(t)$가 어떻게 character화

② $X_n(t) =$ ~~~~~~ (?) (jump direction 초기값)

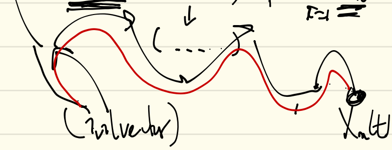
③ 변형 $\frac{X_n(t)}{n}$ 꼴이 나오도록 만들어 본다.

④ 양변에서 $x(t)$를 빼본다.

d-dimensional vector

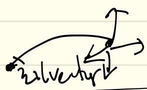
$$X_n(t) = X_n(0) + \sum_{i=1}^{k} \ell_i N_i\left(\int_0^t \underbrace{n\cdot\lambda_i(X_n(s)/n)}_{\text{"rate"}}\,ds\right)$$

(......)



N_i: '독립적' unit-rate poisson process
$i=1,\dots,k$

N_1 N_2 N_k (unit rate poisson clock의 k개)



$$\boxed{\frac{X_n(t)}{n} = \frac{X_n(0)}{n} + \frac{1}{n}\sum_{i=1}^{k}\ell_i N_i\left(\quad\right)}$$

"Centered Poisson process"

Define $\bar{N}_i(t) = N_i(u) - u$

$E(N_i(u)) = u$

centering

$$\frac{X_m(t)}{m} = \frac{X_m(0)}{m} + \frac{1}{m}\left(\sum_{i=1}^{k} e_i \left(\tilde{N}_i\left(\int_0^t m \cdot \lambda_i\left(X_m(s)/m\right) ds\right)\right)\right) + \frac{1}{m}\sum_{i=1}^{k} e_i \int_0^t m \cdot \lambda_i\left(X_m(s)/m\right) ds$$

Let $Y_m(t) = \frac{X_m(t)}{m}$. Recall that $F(x) = \sum_{i=1}^{k} e_i \lambda_i(x)$.

$$Y_m(t) = Y_m(0) + \sum_{i=1}^{k} \frac{e_i}{m}\left(\tilde{N}_i\left(\quad\right)\right) + \int_0^t \underbrace{\sum_{i=1}^{k} e_i \lambda_i\left(X_m(s)/m\right)}_{= F\left(X_m(s)/m\right)} ds$$

$X_m(s)/m = Y_m(s)$

$$\left|\int_0^t f(x)\, dx\right| \leq \int_0^t |f(x)|\, dx$$

기억하시/세요: 우리의 관심사 $|Y_m(t) - x(t)|$

확장하세요: $x(t) = x(0) + \int_0^t F(x(s))\, ds$

$$\boxed{|Y_m(t) - x(t)|} \leq |Y_m(0) - x(0)| + \int_0^t |F(Y_m(s)) - F(x(s))|\, ds + \sum_{i=1}^{k} \frac{|e_i|}{m} \left|\tilde{N}_i\left(\int_0^t\quad\right)\right|$$

$\int_0^t m \cdot \lambda_i(Y_m(s))\, ds$

norm

sup 쓰는다

$$\leq M \int_0^t |Y_m(s) - x(s)|\, ds$$

$\tilde{N}(t) = N(t) - t$

$$\boxed{|f(x)| \leq a + \cdots \int_0^t |f(s)|\, ds + \cdots}$$

assumpt $Y_m(0) \longrightarrow X(0)$ a.s

$$\boxed{|f(x)| \leq ◎ + \int_0^t |f(s)|\, ds + \triangle} \Longrightarrow f(x) \leq \boxed{\quad}$$

$f(x) = \int_0^t$ ~~$f(x) = \int_0^t f(s)\, ds$~~

$f(x) = \int_0^t f(s)\, ds \qquad f(x) = f(x)$

$f(x) = e^{◎}$

Proposition 5.6 (Gronwall's Lemma)

Let u be a bounded real-valued function on $[0, T]$ s.t

$$u(t) \leq a + b \int_0^t u(s)\, ds, \quad \text{for all } t \in [0, T]$$

Then

$$u(t) \leq a e^{bt} \Rightarrow u(T) \leq a e^{bT}$$

$$\sup_{0\le t\le T}\left(|Y_m(t)-x(t)| - M\int_0^t |Y_m(s)-x(s)|\,ds\right) \le |Y_m(0)-x(0)| + \sup_{0\le t\le T}\left(\sum_{i=1}^{k}\frac{|\ell_i|}{m}\left|\bar N_i\left(\int_0^t m\lambda_i(Y_m(s))\,ds\right)\right)$$

for sufficiently large m

$$\le \varepsilon + \sup_{0\le t\le T}\left(\quad\right)$$

$$\Pr\left(\sup_{0\le t\le T}\left(\quad\right) \ge 2\varepsilon\right)$$

$$\le \Pr\left(\sup_{0\le t\le T}\left(\sum_{i=1}^{k}\frac{|\ell_i|}{m}\left|\bar N_i\left(\int_0^t m\lambda_i(Y_m(s))\,ds\right)\right) \ge \varepsilon\right)$$

Proposition 2.29 general version

$$\le 2k\,e^{-mT\bar\lambda\, h\left(\frac{\varepsilon}{kTe\bar\lambda}\right)}$$

ε

$$\Pr\left(\sup_{0\le t\le T}|Y_m(t)-x(t)| \ge \boxed{2\varepsilon e^{MT}}\right)$$

$$\le \Pr\left(\sup_{0\le t\le T}\left(|Y_m(t)-x(t)| - M\int_0^t |Y_m(s)-x(s)|\,ds\right) \ge 2\varepsilon\right)$$

$b = M$

$a = 2\varepsilon$

$$\le 2k\,e^{-mT\bar\lambda h\left(\frac{\varepsilon}{kTe\bar\lambda}\right)}$$

$$\Pr\left(\sup_{0\le t\le T}|Y_m(t)-x(t)| \ge \varepsilon\right) \le \boxed{2k\,e^{-mT\bar\lambda h\left(\quad\right)}}$$

결론

$f(m)$ $\quad \boxed{\sum_m f(m) < \infty}$

$$\Rightarrow \lim_{m\to\infty}\sup_{0\le t\le T}\left|\frac{1}{m}X(t) - x(t)\right| = 0 \quad a.s$$

→ 유한하므로 가능

(Borel-Cantelli Lemma) Let $A_m(\varepsilon) = \{|X_m - X| \ge \varepsilon\}$.

Consider a seq of R.Vs $\{X_m\}$

If $\sum_m P(A_m(\varepsilon)) < \infty$ for all ε, then $X_m \to X$ a.s

a.s $\xrightarrow{\;O\;}$ convergence in probability

$\xleftarrow{\;X\;}$

(Q) under what condition ????

c.i.p $\longrightarrow$ a.s

$\Pr$ $\forall \varepsilon > 0$ $\Pr(A_n(\varepsilon)) \xrightarrow{n\to\infty} 0$ (Convergence in probability)

- $\forall \varepsilon > 0$ $\sum_n P(A_n(\varepsilon)) < \infty \longrightarrow$ a.s 증명 $\sum a_n < \infty \Rightarrow \lim_n a_n = 0$ ($\not\Leftarrow$)

$\downarrow$ $\not\uparrow$ strong 한 조건

$\Pr(A_n(\varepsilon)) \xrightarrow{n\to\infty} 0$

B-C Lemma 의 활용 $X_n \to X$ a.s 증명하는 방법 (sufficient)

① 정의로 이용하기

② $\sum_n P(A_n(\varepsilon)) < \infty \longrightarrow P(A_n(\varepsilon)) \xrightarrow{n\to\infty} 0$ $\to$ a.s

$\Pr(|X_n - X| > \varepsilon) = \frac{1}{n}$ (X)

$\sum \frac{1}{n} < \infty$ (X)

$= \frac{1}{n^2}$ (O) $\sum \frac{1}{n^2} < \infty$

정리 (Gronwall's Lemma 증명

Mode of Convergence

Converge in probability (O) $\to$ a.s (X)

B.C test $\sum P(A_n(\varepsilon)) < \infty$ 가능하게 하는 경우

개념 공부)