

Simulation-based Distributed Coordination Maximization over Networks

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Abstract

In various online/offline multi-agent networked environments, it is very popular that the system can benefit from coordinating actions of two interacting agents at some cost of coordination. In this paper, we first formulate an optimization problem that captures the amount of coordination gain at the cost of node activation over networks. This problem is challenging to solve in a distributed manner, since the target gain is a function of the long-term time portion of the inter-coupled activations of two adjacent nodes, and thus a standard Lagrange duality theory is hard to apply to obtain a distributed decomposition as in the standard Network Utility Maximization. In this paper, we propose three simulation-based distributed algorithms, each having different update rules, all of which require only one-hop message passing and locally-observed information. The key idea for being distributedness is due to a stochastic approximation method that runs a Markov chain simulation incompletely over time, but provably guarantees its convergence to the optimal solution. Next, we provide a game-theoretic framework to interpret our proposed algorithms from a different perspective. We artificially select the payoff function, where the game's Nash equilibrium is asymptotically equal to the socially optimal point, *i.e.*, no Price-of-Anarchy. We show that two stochastically-approximated variants of standard game-learning dynamics overlap with two algorithms developed from the optimization perspective. Finally, we demonstrate our theoretical findings on convergence, optimality, and further features such as a trade-off between efficiency and convergence speed through extensive simulations.

1 Introduction

In many online/offline networking environments, a variety of gains among nodes (or agents) are generated when they make efforts to adjust their states (or actions) with those of others. Two examples include the ones in wireless and social networks. First, in wireless sensor networks with duty cycled node activations for energy saving, each

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sensor node decides to be awake or not over time, which further depends on its neighbors' wake-up state and distance to the node. When two nearby nodes communicate, they are equipped with a robust wireless channel for mutual communication, and thus their coordination (*e.g.*, message exchange) can become more powerful at the cost of energy consumption while they are awake [2,3]. Thus, to achieve the desired coordination gain while turning off redundant sensors, each sensor node should smartly decide to wake up or not, which should often be done in a distributed manner. Second, in online/offline social networks, social relationships and interactions are of critical interest, since the strength of such interactions often determines how the network evolves, *e.g.*, adoption of a new technology or spread of information. For example, when a new technology becomes available, using the social relationships, more coordination gain due to the compatibility of the technology between two individuals is generated, whereas a certain cost due to the adoption of the new technology is incurred, *e.g.*, buying a new software [4–6].

In this paper, we formulate an optimization problem, called *Coordination Gain Maximization* that suitably captures the gain due to peer-to-peer coordinations of connected node pairs and the cost due to individual node activations, as in the following form:

$$\max_{\lambda_i, \lambda_{ij}} \sum_{\text{connected node pair } (i,j)} U_{ij}(\lambda_{ij}) - \sum_{\text{node } i} C_i(\lambda_i), \quad (1)$$

where $U_{ij}(\cdot)$ and $C_i(\cdot)$ are the coordination utility and the node activation cost functions, respectively. Intuitively, λ_{ij} is the long-term time portion when *both* nodes i and j are simultaneously activated and thus coordinated, and λ_i is the long-term time portion when node i is activated. This optimization seems a simple variant of a standard NUM (Network Utility Maximization) [7–9], where it is allowed to easily develop a node-wise distributed algorithm converging to the optimal solution. However, the problem in (1) significantly differs from a standard NUM problem, thus developing a distributed algorithm is far from being trivial. The main challenge of solving this optimization problem lies in the fact that the standard Lagrange duality theory for a distributed decomposition is not possible since the objective function includes the term which is a function of the long-term inter-coupling of the states of a pair of connected nodes, and thus, a separability is not permitted.

In many engineering systems, we often observe the trade-off between efficiency and complexity, where optimal algorithms require extensive message passing or heavy computations, but light-weight approximate algorithms incur suboptimality. Stochastic *simulation-based* algorithms [10, 11] have been investigated in various areas to handle expensive computations in efficient way by using random experimental simulations, in spite of some challenges such as slow convergence time and/or suboptimality of the resulting solution. Our primary goal is to develop a simulation-based distributed coordination decision algorithm that is “efficient”, *i.e.*, hopefully achieving the optimal solution of (1) using random samples of configurations produced by locally-limited message passing. In this work, we formulate an optimization problem of coordination gain maximization over networks by taking a framework of the binary pairwise undirected graphical model, *i.e.*, Ising model [12, 13], to capture pairwise coordinations and nodewise activations of the network, and then propose distributed dynamic

mechanisms which produce the optimal solution of (1). Our main contributions are summarized in what follows:

C1. We first introduce a distributed mechanism, called **CDM(θ)** (Configuration Decision Mechanism), where, by each node’s local state changes based on one-hop message passing, a node activation state of the network is randomly determined in a decentralized manner. This mechanism is governed by a given parameter vector θ that represents the strength of inter-node coordinations and the preference for activation of each node. We illustrate how **CDM** aids in the design of distributed, efficient coordination decision algorithm.

C2. We then propose three simulation-based algorithms, called *Coord-dual*, *Coord-steep*, and *Coord-ind* that provide how to update the parameter vector θ of **CDM** in a distributed manner. We prove that all of three algorithms provably converge to the optimal solution of (1), yet the rationale behind each scheme contains different perspectives of approximation and optimization mechanisms. The key technique towards a distributed operation is to run **CDM(θ)** incompletely over time and exploit *locally-observed* information from random samples to update the parameter θ , which can guarantee the convergence to the optimal solution of (1) on the strength of stochastic approximation theory.

C3. Finally, we take a different angle to understand two algorithms *Coord-steep* and *Coord-ind* using game theory. A game-theoretic framework is one of the powerful tools in the design and analyze the behavior of multi-agent systems, providing valuable insights into various local control rules for agents’ behaviors [14]. In this paper, we design a non-cooperative game with *artificially-selected* payoffs, and show that it has a unique Nash equilibrium which is (asymptotically) equivalent to the socially optimal point, *i.e.*, the optimal solution of (1). We consider popular game dynamics, which we modify with the stochastic approximation technique, and find that those two game dynamics exactly correspond to *Coord-steep* and *Coord-ind*, respectively. We conduct extensive simulations to verify our theoretical findings.

Organization. The rest of the paper is organized as follows. In Section 2, we present a large array of related works. In Section 3, we formulate the coordination gain maximization problem, followed by the analysis of distributed coordination algorithms in Section 4. In Section 5, we provide interpretations from a game-theoretic framework, demonstrate the performance of our algorithms through numerical results in Section 6, and finally conclude in Section 7. All the mathematical proofs are presented in Appendix.

2 Related Work

A variety of benefits from coordinating actions of wireless terminals or users have been widely studied in wireless networks. In the area of wireless sensor networks, various distributed, energy-efficient coordination schemes have been proposed recently, where sensors adaptively select to be coordinators or not, *i.e.*, stay awake and forward sensing data or not, while turning off redundant sensors for energy efficiency. The main interest of this area is scalable, localized, and robust coordination in large-scale environments,

and thus most works have been studied to (i) preserve capacity and connectivity [3], (ii) improve the lifetime of the system and communication latency by using a geo-location information of sensors [15], or (iii) build a self-configuring localization system [2].

In online/offline social networks, coordinating actions, *e.g.*, diffusion /adoption of information, of two (socially) interacting individuals is of importance, since the power of interactions often determines how the network evolves. An importance of a coordination mechanism, *e.g.*, a social structure, for efficient knowledge sharing has been stressed in [16]. There has been a surge of studies about the dynamics of diffusion (*i.e.*, a status of agents) by adopting Ising model in statistical physics [5], epidemic-based models [17], or game-theoretic models [4]. Recently, researchers have studied how coordinated behavior might spread in a network, *i.e.*, game-theoretic diffusion models, and the impact of network structure and/or seeding set on convergence speed [18–21]. Especially, the authors in [18] studied that the noisy best response dynamics converges to the equilibrium, which maximizes the spreading efficiency by choosing an appropriate seed set as in [21].

A large array of work about network utility maximization (NUM) problem have been studied, see [7–9] for surveys. The objective of NUM problem is to maximize a sum of all nodes’ utilities, while not considering any pairwise status, thus separability in the problem often provides a useful dual-based decomposition for an easy development of distributed algorithms. In recent years, the researches on achieving optimality in both throughput and utility in wireless scheduling (in a decentralized manner) have been studied from an optimization perspective [22, 23] as well as from game-theoretic perspective [24–26] for various base-line medium access control protocols. The intuitive idea of these works is that wireless links adaptively adjust access intensities by using local information, *e.g.*, queue-length or empirical service rate, so as to achieve the desired performance.

Our work is based on the importance of the pairwise coordination impacts among individuals, where our main interest is how to find a sequence of node activations (and thus coordinations) in a *decentralized manner* whose long-term status leads to the solution of the problem in (1) that maximizes the network-wide coordination gain at the cost of node activation. Moreover, our work provides new interpretations behind the results obtained from a game-theoretic perspective in the sense that (i) we start from a non-cooperative (ordinal potential) game, followed by the resulting Nash equilibriums’ efficiency (*i.e.*, no Price-of-Anarchy), and (ii) we provide how game-inspired learning dynamics of the game can be connected to the results from an optimization approach.

3 Model and Objective

3.1 System Model

Network model. In this paper, we consider a network $G = (V, E)$ consisting of a set V of nodes and a set $E \subset V \times V$ of edges. With this graphical representation, each node corresponds to an agent in social networks or a sensor node in wireless sensor networks, and each edge corresponds to a physical connectivity or a social relationship between nodes, *i.e.*, $(i, j) \in E$ means that node i and node j are connected and have an

interaction. Note that we study undirected networks where interaction requires mutual consent, *i.e.*, (i, j) is equivalent to (j, i) . Let $\mathcal{N}(i) = \{j \in V : (i, j) \in E\}$ denote the neighbors of node i .

Configuration and coordination scheduling. We consider a continuous time framework. Let $\sigma_i(\tau) \in \{0, 1\}$ indicate whether node i is *active* at time τ or not, *i.e.*, $\sigma_i(\tau) = 1$ means that node i is active at time τ , and 0 otherwise. We say that nodes i and j are (or edge (i, j) is) *coordinated* when $\sigma_i(\tau)\sigma_j(\tau) = 1$. We also denote by $\sigma(\tau) = [\sigma_i(\tau)]_{i \in V}$ a *node configuration* at time τ , and it is clear that a set of possible configurations of the graph G is defined as $\mathcal{I}(G) := \{0, 1\}^{|V|}$. To formally discuss a coordination gain, which we will introduce later, we define a *coordination configuration* as follows:

$$\phi(\sigma) := ([\sigma_i]_{i \in V}, [\sigma_i \sigma_j]_{(i, j) \in E}), \quad (2)$$

which is an augmented configuration vector capturing *both* the activation status of nodes and the coordination status of edges. Then, every coordination configuration belongs to $\Phi(G) := \{0, 1\}^{|V|+|E|}$. Now, a *coordination scheduling* (or simply scheduling) algorithm is a mechanism that selects $\sigma(\tau) \in \mathcal{I}(G)$ (thus a coordination configuration $\phi(\sigma(\tau)) \in \Phi(G)$ is also determined) over time $\tau \in \mathbb{R}_+$.

Coordination region. We now define the maximum achievable *coordination region* (also called coordination capacity region) $\Lambda \subset [0, 1]^{|V|+|E|}$ of the network, which is the convex hull of the feasible coordination configuration set $\Phi(G)$, *i.e.*,

$$\Lambda = \Lambda(G) := \left\{ \sum_{\sigma \in \mathcal{I}(G)} \mu_\sigma \phi(\sigma) : \sum_{\sigma \in \mathcal{I}(G)} \mu_\sigma = 1, \mu_{(\cdot)} \geq 0 \right\}.$$

The intuition of the notion of coordination capacity region comes from the fact that any coordination scheduling algorithm has to choose a node configuration from $\mathcal{I}(G)$ over time (thus a coordination $\phi(\sigma)$ is determined), and w_σ denotes the fraction of time selecting a node configuration σ (and thus a coordination $\phi(\sigma)$). Hence, the long-term (average) time portion of node activation and edge coordination induced by any scheduling algorithm must belong to Λ .

3.2 Problem Description: Coordination Gain Maximization

Objective. We require nodes and edges to control the long-term time portion (or frequency) of activation and coordination close to some boundary of Λ . Specifically, we aim at designing a coordination scheduling algorithm that decides $\sigma(\tau) \in \mathcal{I}(G)$ over time τ so that the long-term time portion of node activation and edge coordination converges to a solution of the following optimization problem:

$$\text{(CG-OPT)} \quad \max_{\lambda \in \Lambda} \sum_{(i, j) \in E} U_{ij}(\lambda_{ij}) - \sum_{i \in V} C_i(\lambda_i). \quad (3)$$

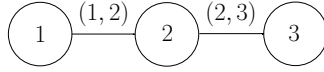


Figure 1: An example line network with 3 nodes and 2 edges, where there are 8 feasible node configurations $\sigma \in \{0, 1\}^3$.

This problem, which we call *coordination gain maximization* problem, captures inter-dependencies among nodes and a trade-off between (edge) coordination utility and (node) activation cost, where $U_{ij} : [0, 1] \rightarrow \mathbb{R}$ is a strictly concave and continuously twice-differentiable coordination utility function of edge $(i, j) \in E$, and $C_i : [0, 1] \rightarrow \mathbb{R}$ is a strictly convex, continuously twice-differentiable activation cost function of node $i \in V$. Then, it is obvious that **CG-OPT** has the unique solution $\lambda^* := ([\lambda_i^*]_{i \in V}, [\lambda_{ij}^*]_{(i,j) \in E})$. The network coordination gain is defined as a total coordination utility subtracted by a total incurring cost. More coordination utility is generated as nodes i and j are coordinated more often, but it also incurs more cost of nodes i and j to be activated.

Example. To illustrate, we provide an example of **CG-OPT** and its solution structure, where we use a line topology with 3 nodes and 2 edges, as depicted in Fig. 1. Now, **CG-OPT** in this example is expressed by:

$$\max_{\lambda} \left[U_{12}(\lambda_{12}) + U_{23}(\lambda_{23}) - \left(C_1(\lambda_1) + C_2(\lambda_2) + C_3(\lambda_3) \right) \right].$$

Let the long-time portion of activation and coordination be characterized by the distribution $\{\pi_{\sigma}\}_{\sigma \in \{0,1\}^3}$, *i.e.*,

$$\begin{aligned} \lambda_1 &= \pi_{(1,0,0)} + \pi_{(1,0,1)} + \pi_{(1,1,0)} + \pi_{(1,1,1)}, \\ \lambda_2 &= \pi_{(0,1,0)} + \pi_{(0,1,1)} + \pi_{(1,1,0)} + \pi_{(1,1,1)}, \\ \lambda_3 &= \pi_{(0,0,1)} + \pi_{(0,1,1)} + \pi_{(1,0,1)} + \pi_{(1,1,1)}, \\ \lambda_{12} &= \pi_{(1,1,0)} + \pi_{(1,1,1)}, \quad \lambda_{23} = \pi_{(0,1,1)} + \pi_{(1,1,1)}. \end{aligned} \quad (4)$$

Note that the total coordination gain is generated according to the long-term coordination time portion of two edges, *i.e.*, $\lambda_{12}, \lambda_{23}$, and the total incurring cost is due to the long-time activation of three nodes, *i.e.*, $\lambda_1, \lambda_2, \lambda_3$.

A smart scheduling is required since each node's activation should be coordinated with its neighboring nodes in order to produce enough gain at the cost of activation. For expositional convenience, let us choose the following utility and cost functions: $U_{12}(x) = U_{23}(x) = \log(x)$, $C_1(x) = C_2(x) = x^2$, and $C_3(x) = 3x^2$, *i.e.*, more cost is incurred for node 3. Now a simple algebra gives the following distributions and the resulting optimal solution:

$$\begin{aligned} \pi_{(0,0,0)}^* &= 0.5, \quad \pi_{(1,1,0)}^* = 0.0915, \quad \pi_{(1,1,1)}^* = 0.4085, \\ (\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_{12}^*, \lambda_{23}^*) &= (0.5, 0.5, 0.4085, 0.5, 0.4085), \end{aligned}$$

where the optimal solution is attained by assigning some probability to the configuration $(1, 1, 0)$ rather than giving a high priority only to $(1, 1, 1)$, with some cost balancing by avoiding the activation of any node, *i.e.*, scheduling $(0, 0, 0)$.

In this work, our goal is to design a *distributed* coordination scheduling algorithm $\{\sigma(\tau)\}_{\tau=0}^{\infty}$ which relies only on local information with one-hop message passing, but converges to the optimal solution of **CG-OPT**, *i.e.*, $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi(\sigma(\tau)) d\tau = \lambda^*$. A lot of challenges may arise, because the developed algorithm should work in an independent manner of the underlying topology and the shape of utility/cost functions, and more importantly the solution should be found in a distributed way, which may entail heavy computations to characterize the long-term time portion of activation and coordination. To overcome these challenges in efficient way, we propose distributed simulation-based algorithms in next section.

4 Distributed Coordination: Algorithm and Analysis

4.1 Configuration Decision Mechanism

We first introduce a parameter $\theta \in \mathbb{R}^{|V|+|E|}$ as:

$$\theta = ([\theta_i]_{i \in V}, [\theta_{ij}]_{(i,j) \in E}). \quad (5)$$

Intuitively, θ_{ij} and θ_i can be interpreted as the *strength* of coordination interaction of edge $(i, j) \in E$ and the *preference* for activation of node $i \in V$, respectively. To capture pairwise interaction of the system, the parameter includes singleton as well as pairwise element, and this parameter will be used as a parameter of algorithms we will design in Section 4.2.

Note that a coordination gain and activation cost of the system would be a function of the long-term time portion of edge coordinations and node activations (*i.e.*, a stationary distribution, say π , of configurations, see the example in (4)), as hinted in **CG-OPT**. This means that it is necessary to develop a time-by-time dynamic mechanism, which, if run for a sufficient amount of time, leads to a certain stationary distribution of configurations for a given θ . In this section, we illustrate how a simple Monte Carlo Markov Chain method may be used as such a time-by-time dynamic mechanism, called **CDM** (Configuration Decision Mechanism). We then identify the optimal distribution over the feasible configurations that maximizes the network-wide coordination gain by producing random samples of configurations via **CDM**. Each algorithm we propose in Section 4.2 produces a sequence of configurations $\{\sigma(\tau)\}_{\tau=0}^{\infty}$ by updating a parameter θ over time so that the resulting long-term activation/coordination rate converges to the optimal solution of **CG-OPT**.

We now describe **CDM**(θ) for a given parameter θ , where every node has a Poisson clock with unit rate and nodes decide a new configuration σ' from a current configuration σ by the procedures as in two steps **S1** and **S2**. Note that Poisson clock of each node leads to the uniform node selection, and for a given graph G , **CDM** decides a configuration over time in a distributed manner with only *one-hop message passing*. In particular, when node i 's clock ticks, it requires to know (i) configuration status of its neighboring nodes, *i.e.*, $\{\sigma_j\}_{j \in \mathcal{N}(i)}$, and (ii) parameter of its neighboring edges, *i.e.*, $\{\theta_{ij}\}_{j \in \mathcal{N}(i)}$, to decide its new configuration σ'_i in **S1**. Then, in **S2**, node i broadcasts its updated configuration σ'_i to all of its neighboring nodes $j \in \mathcal{N}(i)$ for further config-

Configuration Decision Mechanism: **CDM**(θ)

Input: Parameter θ , current configuration $\sigma = [\sigma_i]$.

Output: New configuration $\sigma' = [\sigma'_i]$.

Each node, say i , when its Poisson clock ticks, performs the following:

S1. Node i changes its configuration from σ_i to σ'_i

$$\sigma'_i = \begin{cases} 1, & \text{with probability } \frac{\exp(\theta_i + \sum_{j \in \mathcal{N}(i)} \sigma_j \theta_{ij})}{1 + \exp(\theta_i + \sum_{j \in \mathcal{N}(i)} \theta_{ij} \sigma_j)} \\ 0, & \text{with probability } \frac{1}{1 + \exp(\theta_i + \sum_{j \in \mathcal{N}(i)} \theta_{ij} \sigma_j)} \end{cases} \quad (6)$$

S2. Node i broadcasts its updated σ'_i to all of neighbors in $\mathcal{N}(i)$.

uration decision process. Simply, **CDM**(θ) decides a new configuration mainly based on the status of neighbors with large interaction strength.

One important feature is that **CDM** for a given parameter θ leads to a continuous-time Markov chain $\{\sigma(\tau)\}_{\tau=0}^{\infty}$ achieving the following stationary distribution $p_{\theta} = [p_{\theta, \sigma}]_{\sigma \in \mathcal{I}(G)}$:

$$p_{\theta, \sigma} \propto \exp(\langle \theta, \phi(\sigma) \rangle), \quad \text{for } \sigma \in \mathcal{I}(G), \quad (7)$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ is the inner product of two vectors \mathbf{a} and \mathbf{b} , i.e., $p_{\theta, \sigma} \propto \exp(\sum_{i \in V} \theta_i \sigma_i + \sum_{(i,j) \in E} \theta_{ij} \sigma_i \sigma_j)$. Moreover, $\{\sigma(\tau)\}_{\tau=0}^{\infty}$ is an irreducible, aperiodic, and reversible Markov process [27]. Given the parameter θ , the ergodicity and reversibility of the Markov process imply that the marginal probability of nodes and edges under the stationary distribution p_{θ} , denoted by $\mathbf{s}(\theta) = ([s_i(\theta)]_{i \in V}, [s_{ij}(\theta)]_{(i,j) \in E})$, becomes the long-term time portion of node activation and edge coordination, simply called activation/coordination rate, characterized as¹: for $i \in V$ and $(i, j) \in E$,

$$\begin{aligned} s_i(\theta) &= \mathbb{E}_{p_{\theta}}[\sigma_i] = \sum_{\sigma \in \mathcal{I}(G)} p_{\theta, \sigma} \sigma_i, \\ s_{ij}(\theta) &= \mathbb{E}_{p_{\theta}}[\sigma_i \sigma_j] = \sum_{\sigma \in \mathcal{I}(G)} p_{\theta, \sigma} \sigma_i \sigma_j. \end{aligned} \quad (8)$$

We remark that a graphical model representing the distribution in (7) corresponds to *Ising model* in statistic physics, with *Ising parameter* θ [12], and **CDM**(θ) is a Glauber dynamics over an Ising model under continuous-time setting.

¹We interchangeably use a notation of $\mathbb{E}_{p_{\theta}}[\cdot]$ and $\mathbb{E}_{\theta}[\cdot]$ for the expectation over the distribution p_{θ} .

4.2 Simulation-based Parameter Update Algorithms

In **CG-OPT**, our goal is to find a distribution μ over configurations such that the resulting rate $\mathbb{E}_\mu[\phi(\sigma)]$ becomes the optimal solution of **CG-OPT**. To that end, using **CDM**(θ), we develop three *distributed* simulation-based algorithms that adaptively update the parameter θ over time, which we call **Coord**-algorithms, whose empirical activation/coordination rates from samples generated by **CDM**(θ) asymptotically converge to the optimal solution of **CG-OPT**.

Coord-algorithms: At the start of each frame $t \in \mathbb{Z}_{\geq 0}$

Input: Efficiency parameter $\beta > 0$, smoothing parameter $\alpha \in (0, 1]$, boundary values $\theta^{\min}, \theta^{\max}$.

Output: $\theta[t + 1]$.

Initialize: Set $\theta[0]$ arbitrarily, and $a[0] = 0$.

S1. Each node i sends $\theta_{ij}[t]$ to its neighbor j for all $j \in \mathcal{N}(i)$.

S2. **CDM**($\theta[t]$) is run by each node in a distributed manner, and each node i records the number of its activations $\hat{s}_i[t]$ and its coordinations $\{\hat{s}_{ij}[t]\}_{j \in \mathcal{N}(i)}$ over frame t , and compute the cumulative rate $\bar{s}_i[t], \{\bar{s}_{ij}[t]\}_{j \in \mathcal{N}(i)}$, as in (12).

S3. Each node i updates $\theta_i[t + 1]$ and $\{\theta_{ij}[t + 1]\}_{j \in \mathcal{N}(i)}$ as:

(a) **Coord-dual:**

$$\begin{aligned}\theta_i[t + 1] &= \left[\theta_i[t] + a[t] \left(C_i'^{-1} \left(\frac{-\theta_i[t]}{\beta} \right) - \hat{s}_i[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}, \\ \theta_{ij}[t + 1] &= \left[\theta_{ij}[t] + a[t] \left(U_{ij}'^{-1} \left(\frac{\theta_{ij}[t]}{\beta} \right) - \hat{s}_{ij}[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}.\end{aligned}\quad (9)$$

(b) **Coord-steep:**

$$\begin{aligned}\theta_i[t + 1] &= \left[\theta_i[t] + \alpha \left(-\beta C_i'(\bar{s}_i[t]) - \theta_i[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}, \\ \theta_{ij}[t + 1] &= \left[\theta_{ij}[t] + \alpha \left(\beta U_{ij}'(\bar{s}_{ij}[t]) - \theta_{ij}[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}.\end{aligned}\quad (10)$$

(c) **Coord-ind:**

$$\begin{aligned}\theta_i[t + 1] &= \left[\theta_i[t] + \frac{\alpha}{\beta} \frac{\partial s_i(\theta[t])}{\partial \theta_i} \left(-\beta C_i'(\bar{s}_i[t]) - \theta_i[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}, \\ \theta_{ij}[t + 1] &= \left[\theta_{ij}[t] + \frac{\alpha}{\beta} \frac{\partial s_{ij}(\theta[t])}{\partial \theta_{ij}} \left(\beta U_{ij}'(\bar{s}_{ij}[t]) - \theta_{ij}[t] \right) \right]_{\theta^{\min}}^{\theta^{\max}}.\end{aligned}\quad (11)$$

We now describe *Coord*-algorithms, where $\beta > 0$, $\alpha \in (0, 1]$, $\theta^{\min}, \theta^{\max}$ are the given constants; $[\cdot]_x^y := \max(y, \min(x, \cdot))$; and $a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_+$ is a positive step-size function. In *Coord*-algorithms, time is divided into frames $t = 0, 1, \dots$ of fixed durations T , and each node i updates the parameter θ_i and $\{\theta_{ij}\}_{j \in \mathcal{N}(i)}$ ² following one of three schemes: (a) *Coord-dual*, (b) *Coord-steep* and (c) *Coord-ind*. In **S1**, at the beginning of each frame t , node i sends each of $\{\theta_{ij}[t]\}_{j \in \mathcal{N}(i)}$ to each j of its neighbors³. Then in **S2**, $\text{CDM}(\theta[t])$ runs in a distributed manner, leading to the local computation of *instant* and *cumulative* activation/coordination rates at and until the frame t , denoted by $\hat{s}[t]$ and $\bar{s}[t]$, respectively, *i.e.*,

$$\hat{s}[t] = \frac{1}{T} \int_{tT}^{(t+1)T} \phi(\sigma(\tau)) d\tau, \quad \bar{s}[t] = \frac{1}{t} \sum_{m=0}^t \hat{s}[m], \quad (12)$$

where both empirical rates are *locally-computed*. In **S3**, each scheme utilizes either of the computed empirical rates: $\hat{s}[t]$ for *Coord-dual* and $\bar{s}[t]$ for *Coord-steep*, *Coord-ind*. Note that $\nabla s(\theta[t])$ is also locally obtained (see Appendix for a detailed form of $\nabla s(\cdot)$), thus all of *Coord*-algorithms are run in a distributed manner.

4.3 Rationale behind *Coord*-algorithms

We now explain the rationale behind each scheme of *Coord*-algorithms that contains different perspectives of approximation and optimization mechanisms.

(a) *Coord-dual*: Note that **CG-OPT** in (3) can be written as:

$$\max_{\mu \in \mathcal{M}} \mathcal{F}(\mu) := \sum_{(i,j) \in E} U_{ij}(\mathbb{E}_{\mu}[\sigma_i \sigma_j]) - \sum_{i \in V} C_i(\mathbb{E}_{\mu}[\sigma_i]), \quad (13)$$

where \mathcal{M} is a set of all probability measures over the feasible configurations $\mathcal{I}(G)$. From this, we consider the following variant **A-CG-OPT** (parameterized by $\beta > 0$) of **CG-OPT**:

$$\begin{aligned} & \text{(A-CG-OPT)} \\ & \max \quad \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}) - \sum_{i \in V} C_i(\lambda_i) + \frac{1}{\beta} H(\mu) \\ & \text{over} \quad \mu \in \mathcal{M}, \lambda \in [0, 1]^{|V|+|E|} \\ & \text{subject to} \quad \lambda_i = \mathbb{E}_{\mu}[\sigma_i], \quad \forall i \in V, \\ & \quad \quad \quad \lambda_{ij} = \mathbb{E}_{\mu}[\sigma_i \sigma_j], \quad \forall (i, j) \in E, \end{aligned} \quad (14)$$

where $H(\mu) = -\sum_{\sigma \in \mathcal{I}(G)} \mu_{\sigma} \log \mu_{\sigma}$ is the *entropy* of μ .

²In practice, each node i may have additional independent controllers for its neighboring edges (i, j) of $j \in \mathcal{N}(i)$. Either node i or j may have the control authority of edge (i, j) following some arbitrary rule.

³We implicitly assume that information exchange in **S1** of *Coord*-algorithms and **S2** in $\text{CDM}(\theta)$ can be done by out-of-band signaling, *i.e.*, a separate control channel.

Note that, compared to **CG-OPT**, **A-CG-OPT** has additional term $\frac{1}{\beta}H(\boldsymbol{\mu})$ in its objective function. Since the entropy is bounded, *i.e.*, $|H(\boldsymbol{\mu})| \leq \log |\mathcal{I}(G)|$, a solution of **A-CG-OPT**, say $(\boldsymbol{\mu}^\circ, \boldsymbol{\lambda}^\circ)$, becomes arbitrarily closer to that of **CG-OPT** for large β , which we call an efficiency parameter. Moreover, the entropy term leads to a distributed algorithm achieving the solution of **A-CG-OPT** in following way.

Regarding the parameter $\boldsymbol{\theta}$ as a dual variable of the Lagrangian of **A-CG-OPT**, its dual problem is simply represented as $\min_{\boldsymbol{\theta}} \mathcal{D}(\boldsymbol{\theta})$ (see for a detailed form of $\mathcal{D}(\cdot)$). Then, the steepest descent method to solve this dual problem with the direction $\mathbf{d}[t] = -\nabla \mathcal{D}(\boldsymbol{\theta}[t])$ and step-size $a[t]$ is given by $\boldsymbol{\theta}[t+1] = \boldsymbol{\theta}[t] + a[t] \cdot \mathbf{d}[t]$, *i.e.*,

$$\begin{aligned}\theta_i[t+1] &= \theta_i[t] + a[t] \left(C_i'^{-1} \left(\frac{-\theta_i[t]}{\beta} \right) - s_i(\boldsymbol{\theta}[t]) \right), \\ \theta_{ij}[t+1] &= \theta_{ij}[t] + a[t] \left(U_{ij}'^{-1} \left(\frac{\theta_{ij}[t]}{\beta} \right) - s_{ij}(\boldsymbol{\theta}[t]) \right).\end{aligned}\quad (15)$$

We highlight that **Coord-dual** in (9) is a distributed implementation of (15), where the key idea is to use (i) the instant rate $\hat{s}_i[t], \hat{s}_{ij}[t]$ from the current samples with (ii) diminishing step-size $a[t]$ (*i.e.*, satisfying (24)), instead of computing the exact rate $s_i(\boldsymbol{\theta}[t]), s_{ij}(\boldsymbol{\theta}[t])$. Recall that computing the service rate directly requires information of all other nodes and edges, and measuring the service rate (*i.e.*, the marginal probability in the Markov chain induced by $\mathbf{CDM}(\boldsymbol{\theta}[t])$) requires a *mixing time* to reach the stationary distribution from a large number of samples. The proof of convergence and optimality of **Coord-dual** using $\hat{s}[t]$ with $a[t]$ is due to the stochastic approximation technique [28–30], as presented in Section 4.4.

(b) Coord-steep: Taking the different perspective of **CG-OPT**, at any time, we sample the configuration via **CDM**, offering the steepest ascent direction for $\mathcal{F}(\boldsymbol{\mu})$ in (13). Among feasible coordinates (*i.e.*, elements) of $\boldsymbol{\mu} = [\mu_\sigma]_{\sigma \in \mathcal{I}(G)}$, the steepest coordinate ascent method to solve (13) deduces to select a configuration $\boldsymbol{\sigma}_*$ according to the rule⁴:

$$\boldsymbol{\sigma}_* = \arg \max_{\boldsymbol{\sigma} \in \mathcal{I}(G)} \nabla_{\boldsymbol{\sigma}} \mathcal{F}(\boldsymbol{\mu}),$$

where from (13),

$$\frac{\partial \mathcal{F}(\boldsymbol{\mu})}{\partial \mu_\sigma} = \sum_{(i,j) \in E} \sigma_i \sigma_j U_{ij}'(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j]) - \sum_{i \in V} \sigma_i C_i'(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i]). \quad (16)$$

Then, sampling configurations from the distribution that concentrates on $\boldsymbol{\sigma}_*$, *e.g.*, a distribution (parameterized by β), say $\bar{\boldsymbol{\mu}}$, such that $\bar{\boldsymbol{\mu}}_\sigma \propto \exp(\beta \cdot \nabla_{\boldsymbol{\sigma}} \mathcal{F}(\bar{\boldsymbol{\mu}}))$, approximates what the perfect steepest ascent method would do. Therefore, from (7) and (16), the steepest ascent method solving (13) is approximated via **CDM**($\boldsymbol{\theta}$) by setting $\boldsymbol{\theta}$ as follows:

$$\theta_i = -\beta \cdot C_i'(\mathbb{E}_{\boldsymbol{\theta}}[\sigma_i]), \quad \theta_{ij} = \beta \cdot U_{ij}'(\mathbb{E}_{\boldsymbol{\theta}}[\sigma_i \sigma_j]). \quad (17)$$

⁴A partial derivative of a function f at the point \mathbf{x} with respect to the i -th variable x_i is denoted by $\frac{\partial f(\mathbf{x})}{\partial x_i}$, or simply $\nabla_i f(\mathbf{x})$.

Note that the target parameter θ is a fixed point of (17), and its distribution p_θ depends on the marginal probability $\mathbb{E}_\theta[\phi(\sigma)]$, which may evolve over time. Now, a fixed point iteration method of (17) is given by

$$\theta_i[t+1] = -\beta C'_i(\mathbb{E}_{\theta[t]}[\sigma_i]), \quad \theta_{ij}[t+1] = \beta U'_{ij}(\mathbb{E}_{\theta[t]}[\sigma_i \sigma_j]).$$

To smooth out the effect of random movements of the marginal probability in (17) and take a fixed point in a limit, we consider an *exponential moving average* (EMA) with a constant smoothing parameter $\alpha \in (0, 1]$ as follows:

$$\begin{aligned} \theta_i[t+1] &= \alpha \left(-\beta C'_i(\mathbb{E}_{\theta[t]}[\sigma_i]) \right) + (1-\alpha)\theta_i[t], \\ \theta_{ij}[t+1] &= \alpha \left(\beta U'_{ij}(\mathbb{E}_{\theta[t]}[\sigma_i \sigma_j]) \right) + (1-\alpha)\theta_{ij}[t]. \end{aligned} \quad (18)$$

The key rationale of **Coord-steep** in (10) towards a distributed operation of (18) is to use the cumulative rate $\bar{s}[t]$ instead of $\mathbb{E}_{\theta[t]}[\phi(\sigma)]$, which can guarantee the convergence to the optimal point (for large β), again, due to the stochastic approximation technique. In particular, from (12), we have

$$\bar{s}[t] = \bar{s}[t-1] - \frac{1}{t}(\bar{s}[t-1] - \hat{s}[t]), \quad t \in \mathbb{Z}_{\geq 0}, \quad (19)$$

thus the use of cumulative rate $\bar{s}[t]$ has a similar effect of exploiting instant rate $\hat{s}[t]$ with $\frac{1}{t}$ step-size (*i.e.*, satisfying (24)), as in **Coord-dual**.

(c) **Coord-ind**: A simple, myopic approach for a distributed operation to solve **CG-OPT** would be to decompose its objective into node/edge-wise local optimization problems, *i.e.*, node i minimizes its cost and edge (i, j) maximizes its utility. Considering long-term activation/coordination rates under the stationary distribution p_θ , we associate each component of the parameter θ with each local problem, *i.e.*, $\min_{\theta_i} C_i(s_i(\theta))$ and $\max_{\theta_{ij}} U_{ij}(s_{ij}(\theta))$. However, such a myopic approach does not guarantee to achieve the optimal solution of **CG-OPT** due to the inter-coupling from θ in the objective functions. To reflect this inter-coupling among nodes and edges, we design the following problem **A-IND-OPT**⁵ with new objective function, denoted by $\Psi_i(\theta_i)$ for node i and $\Psi_{ij}(\theta_{ij})$ for edge (i, j) , where the key part lies in including *artificially-designed* penalty terms in $\Psi_i(\theta_i)$ and $\Psi_{ij}(\theta_{ij})$.

$$\begin{aligned} \text{(A-IND-OPT)} \quad & \max_{\theta_i \in \mathbb{R}} \Psi_i(\theta_i), \quad i \in V, \\ & \max_{\theta_{ij} \in \mathbb{R}} \Psi_{ij}(\theta_{ij}), \quad (i, j) \in E, \\ \text{where} \quad & \Psi_i(\theta_i) = -C_i(s_i(\theta)) - \frac{1}{\beta} \int_{-\infty}^{\theta_i} x s'_i(x, \theta_{-i}) dx, \\ & \Psi_{ij}(\theta_{ij}) = U_{ij}(s_{ij}(\theta)) - \frac{1}{\beta} \int_{-\infty}^{\theta_{ij}} x s'_{ij}(x, \theta_{-ij}) dx. \end{aligned} \quad (20)$$

⁵We denote the parameter vector for all other components except node i by θ_{-i} , *i.e.*, $\theta = (\theta_i, \theta_{-i})$, and similarly θ_{-ij} for edge (i, j) .

The basic rationale of **A-IND-OPT** is that each node and edge chooses its own parameter by considering only its own cost or utility, yet it might lead to the network-wide optimal status when imposing an appropriate amount of penalty, *i.e.*, the second term of (20). The form of penalty (parameterized by β) is of critical importance to achieve the global optimal solution. This individually strategic form is well understood by a game-theoretic perspective, as presented in Section 5.1.

Now, the steepest ascent method to solve **A-IND-OPT** with the direction $d_i[t] = \nabla \Psi_i(\theta_i[t])$, $d_{ij}[t] = \nabla \Psi_{ij}(\theta_{ij}[t])$ (see Appendix for the full derivation) and the step-size $a[t]$ is given by $\boldsymbol{\theta}[t+1] = \boldsymbol{\theta}[t] + a[t] \cdot \mathbf{d}[t]$, *i.e.*,

$$\begin{aligned}\theta_i[t+1] &= \theta_i[t] + \frac{a[t]}{\beta} \frac{\partial s_i(\boldsymbol{\theta}[t])}{\partial \theta_i} \left(-\beta C'_i(s_i(\boldsymbol{\theta}[t])) - \theta_i[t] \right), \\ \theta_{ij}[t+1] &= \theta_{ij}[t] + \frac{a[t]}{\beta} \frac{\partial s_{ij}(\boldsymbol{\theta}[t])}{\partial \theta_{ij}} \left(\beta U'_{ij}(s_{ij}(\boldsymbol{\theta}[t])) - \theta_{ij}[t] \right).\end{aligned}\quad (21)$$

Similarly to two earlier schemes, the key technique of **Coord-ind** is a stochastic approximation, *i.e.*, exploiting the cumulative rate $\bar{s}[t]$ from samples, instead of computing the exact $\mathbf{s}(\boldsymbol{\theta}[t])$, $\nabla \mathbf{s}(\boldsymbol{\theta}[t])$. The role of adopting the cumulative rate $\bar{s}[t]$ towards convergence to optimality can be clearly seen by introducing the following alternative sequence, say $\{\boldsymbol{\eta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$, which will be shown to track the sequence $\{\boldsymbol{\theta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of (21) using $\bar{s}[t]$ with a constant step-size $a[t] = \alpha \in (0, 1]$ (see Appendix for details), defined as:

$$\eta_i[t+1] = -\beta C'_i(\bar{s}_i[t]), \quad \eta_{ij}[t+1] = \beta U'_{ij}(\bar{s}_{ij}[t]). \quad (22)$$

From (19), the iterative update rule of the alternative sequence is represented as follows: for large t ,

$$\begin{aligned}\eta_i[t+1] &= \eta_i[t] + \frac{1}{t} g_i(\eta_i[t]) \left(C_i'^{-1} \left(\frac{-\eta_i[t]}{\beta} \right) - \hat{s}_i[t] \right), \\ \eta_{ij}[t+1] &= \eta_{ij}[t] + \frac{1}{t} g_{ij}(\eta_{ij}[t]) \left(U_{ij}'^{-1} \left(\frac{\eta_{ij}[t]}{\beta} \right) - \hat{s}_i[t] \right),\end{aligned}$$

where

$$g_i(x) = \beta C_i'' \left(C_i'^{-1} \left(\frac{-x}{\beta} \right) \right), \quad g_{ij}(x) = -\beta U_{ij}'' \left(U_{ij}'^{-1} \left(\frac{x}{\beta} \right) \right). \quad (23)$$

Note that the alternative process (22) has an effect of exploiting $\hat{\mathbf{s}}[t]$ with diminishing step-size $\frac{1}{t}$, as in **Coord-dual**.

4.4 Convergence and Optimality Analysis

For provable convergence analysis, we first make the following assumption, implying that we choose θ^{\min} and θ^{\max} , such that the interval $[\theta^{\min}, \theta^{\max}]$ is large enough to include the optimal solution of **A-CG-OPT**⁶.

⁶The explicit values of θ^{\min} and θ^{\max} can be also computable as in [31].

(A1) If $\theta^0 \in \mathbb{R}^{|V|+|E|}$ solves for all $i \in V$ and $(i, j) \in E$,

$$\theta_i^0 = -\beta C_i'(\mathbb{E}_{\theta^0}[\sigma_i]), \quad \theta_{ij}^0 = \beta U_{ij}'(\mathbb{E}_{\theta^0}[\sigma_i \sigma_j]),$$

then $\theta^{\min} \leq \theta_i^0 \leq \theta^{\max}$ and $\theta^{\min} \leq \theta_{ij}^0 \leq \theta^{\max}$. Note that, for example, if $U_{ij}'(0) < \infty$, then (A1) for θ_{ij}^0 is satisfied when $\theta^{\min} \leq \beta U_{ij}'(1)$ and $\theta^{\max} \geq \beta U_{ij}'(0)$.

Now, the next theorem is our main result, which states the convergence of **Coord**-algorithms to a point arbitrarily close to the optimal solution of **CG-OPT**, under some mild conditions.

Theorem 1 (Convergence/Optimality of **Coord**-algorithms).

(i) Convergence. Under (A1), for strictly concave/convex, continuously twice-differentiable utility/cost functions, choose a step-size function $a[\cdot]$ in **Coord-dual** satisfying

$$\sum_t a[t] = \infty, \quad \sum_t a[t]^2 < \infty. \quad (24)$$

Then, for any initial condition $\theta[0]$, under all **Coord**-algorithms, $\theta[t]$ and corresponding $\bar{s}[t]$ (from (12)) converges to $(\theta^\circ, \lambda^\circ)$, i.e.,

$$\lim_{t \rightarrow \infty} \theta[t] = \theta^\circ \quad \text{and} \quad \lim_{t \rightarrow \infty} \bar{s}[t] = \lambda^\circ, \quad \text{almost surely,}$$

where $(\theta^\circ, \lambda^\circ)$ is such that $(p_{\theta^\circ}, \lambda^\circ)$ attains the (unique) solution of **A-CG-OPT** in (14) (over μ and λ).

(ii) Optimality. Furthermore, **Coord**-algorithms approximately solve **CG-OPT** in the following sense:

$$\begin{aligned} & \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^\circ) - \sum_{i \in V} C_i(\lambda_i^\circ) \geq \\ & \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^*) - \sum_{i \in V} C_i(\lambda_i^*) - \frac{\log |\mathcal{I}(G)|}{\beta}, \end{aligned} \quad (25)$$

where λ^* is the optimal solution of **CG-OPT** in (3).

The proof of Theorem 1 is presented in Appendix, but we briefly provide the proof sketch for readers' convenience. Each scheme of **Coord**-algorithms is interpreted as a stochastic approximation procedure with controlled Markov noise, and a main technical challenge lies in handling a non-trivial coupling between Markov process of **CDM**(θ) and parameter θ updates. Simply, a provable convergence is guaranteed on the strength of stochastic approximation theory, in that we intuitively expect that by exploiting (i) instant rate $\hat{s}[\cdot]$ with diminishing step-size in (24) or (ii) cumulative rate $\bar{s}[\cdot]$ which has an effect of $\frac{1}{t}$ step-size (i.e., satisfying (24)), the speed of variations of the parameter θ tends to zero after sufficiently long time. Thus, its limiting behavior can be understood by ordinary differential equations (ODE). We highlight that we

adopt diminishing step-size in (24), following the standard ODE approaches in stochastic approximation theory as in [28, 32, 33] and references therein, to provide provable convergence. The additional challenge dealing with *Coord-steep* and *Coord-ind* (not existing for *Coord-dual*) is that they have higher-order temporal dependencies in their updating rules, *i.e.*, use the current parameter $\theta[t]$ directly when obtaining the next parameter $\theta[t + 1]$. To handle this issue, we define ‘alternative’ process (see $\{\rho[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ and $\{\eta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ in Appendix) and argue its convergence under the relation to that of the original process $\{\theta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$.

5 New Interpretations via Game Theory

In Section 4, we develop three simulation-based algorithms that adaptively update the parameter θ in a distributed manner, but result in the optimal solution of **CG-OPT**. The rationale behind each scheme follows the framework of distributed optimization. In this section, we take a different angle to reverse engineer two algorithms *Coord-steep* and *Coord-ind* in other framework, which is the game-theoretic one. As a background, game theory has been emerged as a powerful tool not only to analyze the rational behavior of competitive multi-agent systems (*i.e.*, just optimizing a local objective), but also to control the local behavior of each agent, see *e.g.*, [14]. In such a framework, it is aimed that a game is designed with an *artificially-selected* payoff function so that local decisions of agents result in a system-wide desirable solution such as an unique, fair or socially-optimal point. Moreover, a game-theoretic approach provides valuable insights into the design of various robust local control rules through (distributed) game dynamics, whereas the standard centralized optimization framework can not directly consider the interactions among agents. In this section, inheriting such a philosophy of the game-theoretic framework for distributed optimization, we establish the desirable properties of equilibrium from a well-designed non-cooperative game and present that *Coord-steep* and *Coord-ind* correspond to the stochastically-approximated variants of two popular game-learning dynamics.

5.1 CoordGain(β) Game

We first design a non-cooperative game, denoted by CoordGain(β) with $\beta > 0$.

CoordGain(β)

- (i) **Players.** Each node $i \in V$ and each edge $(i, j) \in E$ acts as a player. Let $N = V \cup E$ denote the set of players, and thus $n \in N$ can be either a node $i \in V$ or an edge $(i, j) \in E$ ⁷.
- (ii) **Strategy.** Each player n has a parameter $\theta_n \in \mathbb{R}$ as its own strategy. We denote the strategy profile of entire players by $\theta = [\theta_n]_{n \in N} = ([\theta_i]_{i \in V}, [\theta_{ij}]_{(i,j) \in E}) \in \mathbb{R}^{|N|}$.

⁷We interchangeably use a notation of ij and (i, j) for an edge player.

- (iii) **Payoff.** The payoff function of player $n \in N$, denoted by $\Psi_n(\theta_n, \theta_{-n}) : \mathbb{R}^{|N|} \mapsto \mathbb{R}$, is designed to be a net-coordination utility (or net-activation cost) with incurring *penalty* function $V_n(\cdot)$ resulting from coordination:

$$\begin{aligned}\Psi_i(\theta_i, \theta_{-i}) &= -C_i(s_i(\boldsymbol{\theta})) - \frac{1}{\beta} V_i(\theta_i, \theta_{-i}), \\ \Psi_{ij}(\theta_{ij}, \theta_{-ij}) &= U_{ij}(s_{ij}(\boldsymbol{\theta})) - \frac{1}{\beta} V_{ij}(\theta_{ij}, \theta_{-ij}),\end{aligned}\quad (26)$$

$$\begin{aligned}\text{where } V_i(\theta_i, \theta_{-i}) &= \int_{-\infty}^{\theta_i} x s'_i(x, \theta_{-i}) dx, \\ V_{ij}(\theta_{ij}, \theta_{-ij}) &= \int_{-\infty}^{\theta_{ij}} x s'_{ij}(x, \theta_{-ij}) dx.\end{aligned}\quad (27)$$

Note that a player n 's payoff $\Psi_n(\cdot)$ ⁸ is determined by how aggressively other players are activated/coordinated as well as how itself does. The parameter β quantifies *penalty level* in the players' payoffs, and we realize that it balances the trade-off between the quality of equilibria and the convergence speed to the equilibria under game dynamics (see Theorem 2).

To achieve our goal of obtaining good equilibria and a provable transfer to distributed game dynamics converging to an equilibrium, the choice of penalty function $V_n(\cdot)$ is of crucial importance. Our choice of penalty function (27) captures following two features.

First, it appropriately measures each player's impact of excessive strategy on other players. One naïve choice of penalty to be imposed by a player n may be $V_n(\boldsymbol{\theta}) = \theta_n \times s_n(\theta_n, \theta_{-n})$, which is proportional to the current strategy θ_n multiplied by its achieved long-term gain $s_n(\boldsymbol{\theta})$, yet it is unclear that this penalty provides a provable framework of equilibrium analysis. On the other hand, our design of penalty function considers the expected strategy value $\mathbb{E}[\Theta_n]$ which depends on the *relative increasing speed* of one's rate in the interval $(-\infty, \theta_n)$, by letting $\Theta_n \in [-\infty, \theta_n]$ denote a continuous random variable with the density function $f_{\Theta_n}(x) = \frac{1}{s_n(\theta_n, \theta_{-n})} \frac{\partial s_n(x, \theta_{-n})}{\partial x}$ so that the penalty function is represented as $V_n(\boldsymbol{\theta}) = \mathbb{E}[\Theta_n] \times s_n(\theta_n, \theta_{-n})$.

Second, the penalty function (27) is a function of self-strategy and its marginal distribution, not the individual strategy values or payoffs of others. From simple algebra of (8), it is shown to be structured in terms of local information: $V_n(\boldsymbol{\theta}) = \theta_n s_n(\boldsymbol{\theta}) + \ln(1 - s_n(\boldsymbol{\theta}))$. Since $s_n(\cdot)$ can be measured in the midst of playing a player's own strategy, *e.g.*, $\hat{s}_n(\cdot)$, via **CDM** with one-hop message passing, best response or payoff gradient of our game can be locally estimated. This feature enables us to develop distributed game dynamics, which indeed corresponds to **Coord-steep** and **Coord-ind** (see Section 5.3).

⁸We use the notation Ψ_i, Ψ_{ij} in both (20) and (26) for notational simplicity, since they obviously have the same detailed form.

5.2 Equilibrium Analysis

We first present popular notions: Nash equilibrium and Price-of-Anarchy in game theory:

Definition 1. *In the $\text{CoordGain}(\beta)$,*

(i) *a strategy profile θ^{NE} is a Nash equilibrium (NE) if*

$$\Psi_n(\theta_n^{\text{NE}}, \theta_{-n}^{\text{NE}}) \geq \Psi_n(\theta_n, \theta_{-n}^{\text{NE}}), \quad \forall \theta_n \in \mathbb{R}, \forall n \in N.$$

(ii) *a Price-of-Anarchy (PoA) is*

$$\text{PoA} = \frac{\max_{\theta} \sum_{(i,j) \in E} U_{ij}(s_{ij}(\theta)) - \sum_{i \in V} C_i(s_i(\theta))}{\min_{\theta^{\text{NE}}} \sum_{(i,j) \in E} U_{ij}(s_{ij}(\theta)) - \sum_{i \in V} C_i(s_i(\theta))}.$$

Furthermore, we say that a NE θ^{NE} (if exists) of the game is *non-trivial*, if players' activation/coordination rate at equilibrium $s(\theta^{\text{NE}})$ is positive, and *trivial* otherwise. The PoA indicates the ratio between the social optimum and the worst equilibrium of the game, and we say *no PoA* if $\text{PoA} = 1$. We now present our main results on the equilibrium analysis.

Theorem 2 (Uniqueness and PoA). *In the $\text{CoordGain}(\beta)$,*

(i) Uniqueness. *for any $\beta > 0$, there exists a unique non-trivial NE θ^{NE} .*

(ii) Price-of-Anarchy. *$(p_{\theta^{\text{NE}}}, s(\theta^{\text{NE}}))$ attains the optimal solution of **A-CG-OPT**, and thus $\lim_{\beta \rightarrow \infty} \text{PoA} = 1$.*

The proof of Theorem 2 is presented in Appendix. We prove that our game is an *ordinal potential game*, where the potential function corresponds to the dual function of **A-CG-OPT**. It implies that our game has a unique non-trivial NE θ^{NE} , in particular, the solution of **A-CG-OPT** is attained at $(p_{\theta^{\text{NE}}}, s(\theta^{\text{NE}}))$, *i.e.*, $\theta^{\text{NE}} = \theta^o$. Therefore, there is asymptotically no PoA in our game, *i.e.*, the aggregate coordination gain at the unique non-trivial NE becomes arbitrarily close to the social optimum by choosing sufficiently large β .

5.3 Reverse Engineering of *Coord-steep* and *Coord-ind*

Best response dynamics. The most popular dynamics is the *best response (BR) dynamics* that each player chooses its best strategy, given the strategy (at the previous frame) of all other players, *i.e.*, at frame t ,

$$\theta_n[t+1] = \text{BR}_n(\theta_{-n}[t]) := \arg \max_{\theta_n \in \mathbb{R}} \Psi_n(\theta_n, \theta_{-n}[t]),$$

which leads to a fixed point of the following function in $\text{CoordGain}(\beta)$: $\forall i \in V, \forall (i, j) \in E$,

$$\theta_i[t+1] = -\beta C'_i(s_i(\theta_i[t+1], \theta_{-i}[t])),$$

$$\theta_{ij}[t+1] = \beta U'_{ij} \left(s_{ij}(\theta_{ij}[t+1], \theta_{-ij}[t]) \right). \quad (28)$$

Jacobi dynamics. The second dynamics is *Jacobi dynamics*, whose idea is to adjust each player’s strategy gradually towards its best response strategy, *i.e.*, at frame t ,

$$\theta_n[t+1] = \theta_n[t] + \alpha \cdot (\text{BR}_n(\theta_{-n}[t]) - \theta_n[t]),$$

where $\alpha \in (0, 1]$ is a smoothing parameter⁹. The smoothing parameter captures how accurately the dynamics follows the BR dynamics, where $\alpha = 1$ corresponds to the BR dynamics. From (18) and (28), we can verify that a variant of Jacobi dynamics of our game (BR dynamics as a special case) which exploits cumulative rate $\bar{s}[t]$ instead of $s(\theta[t])$ at every frame is indeed equivalent to **Coord-steep**, *i.e.*, approximated (parameterized by β) steepest ascent method of **CG-OPT**.

Gradient dynamics. Finally, we investigate the *gradient dynamics* [34] that each player n first determines the gradient of its payoff (26), $\nabla \Psi_n(\theta)$, then updates its strategy in that direction with a constant step-size $\alpha \in (0, 1]$, *i.e.*, at frame t ,

$$\theta_n[t+1] = \theta_n[t] + \alpha \cdot \nabla \Psi_n(\theta_n[t]).$$

The interpretation of the gradient dynamics from an economic perspective is that if the marginal coordination utility of an edge (i, j) exceeds the marginal penalty, *i.e.*, $\nabla \Psi_{ij}(\theta) > 0$, its strategy is increased to achieve more coordination gain, and if $\nabla \Psi_{ij}(\theta) < 0$, its strategy is decreased to reduce the penalty. From the objective function (20) and our payoff function (26), we can verify that **Coord-ind** is equivalent to a variant of gradient dynamics of our game, which exploits $\bar{s}[t]$ from samples instead of computing the exact $s(\theta[t])$, $\nabla s(\theta[t])$.

To summarize, **Coord-steep**, **Coord-ind** are stochastically-approximated variants of Jacobi dynamics and gradient dynamics of **CoordGain**(β), respectively. Theorem 1 states that those dynamics converge to the unique non-trivial NE. Note that this is a new feature in our work, not prevalent in general game-theoretic approaches for distributed optimization, *i.e.*, there exists no generalized distributed dynamics converging to a NE (even it exists) due to a lack of information, in a broad class of games [35].

6 Numerical Results

In this section, we carry out numerical experiments to assess our analytical findings of **Coord**-algorithms by considering networks with various topologies and cost functions.

Setup. In this paper, we consider “basic” topologies to show that our **Coord**-algorithms converge to the accurate solution, and a random topology that is regarded as a collection of such basic topologies for more general results. The network topologies under which our results are presented here are star, complete, and random graphs. For numerical results, we consider proportional fairness across edges for coordination utility, *i.e.*,

⁹Jacobi dynamics generally makes a smoother move than the BR dynamics, where a small smoothing parameter plays the role of compensating for the instability of the BR dynamics, see [24].

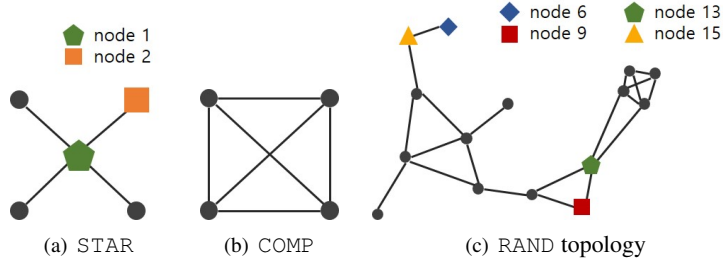


Figure 2: Network topologies

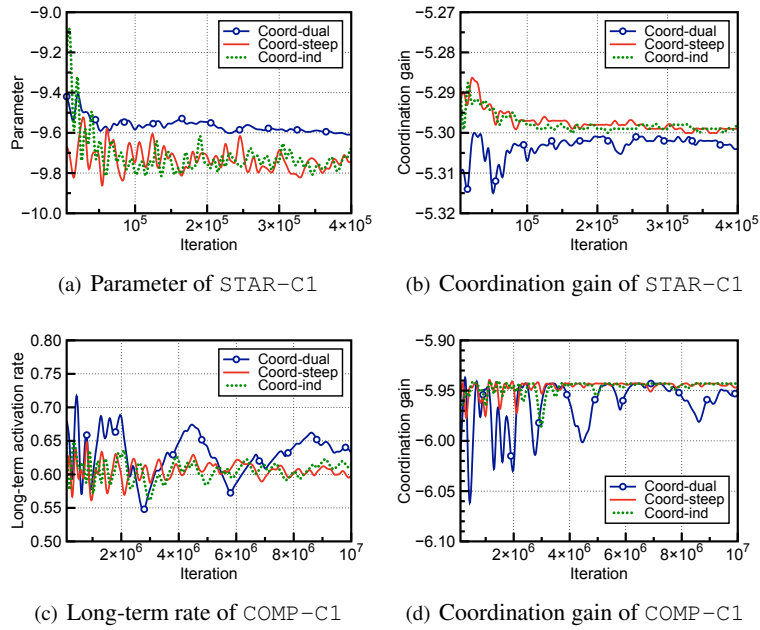


Figure 3: Convergence of parameter, coordination gain, and long-term rate to optimality on STAR-C1 and COMP-C1.

$U_{ij}(x) = \log(x)$ for all edges $(i, j) \in E$, and consider two cost functions for nodes: (C1) $C_i(x) = 2x^2$ and (C2) $C_i(x) = \frac{1}{1-x}$ for all nodes $i \in V$, as classified into the following 4 topologies.

- STAR-C1: Star graph of 5 nodes with (C1)
- COMP-C1: Complete graph of 4 nodes with (C1)
- RAND-C1: Random graph of 15 nodes, 21 edges with (C1)
- RAND-C2: Random graph of 15 nodes, 21 edges with (C2)

The above topologies are depicted in Fig. 2: Fig. 2(a) for STAR-C1, Fig. 2(b)

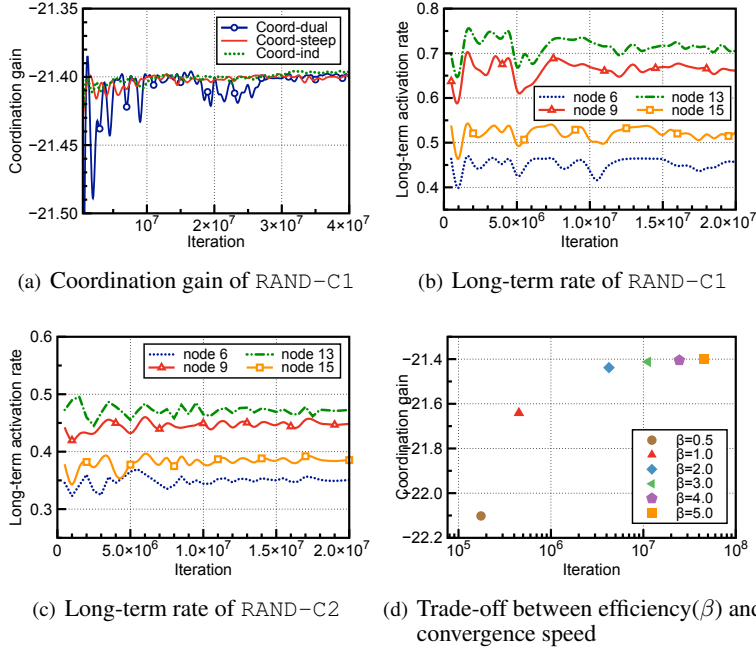


Figure 4: Convergence of coordination gain, long-term rate, and trade-off on RAND-C1 and RAND-C2.

for COMP-C1, and Fig. 2(c) for RAND-C1, RAND-C2. Moreover, for a fixed frame duration $T = 10$, we choose a step-size function $a[t] = 3/t$ for **Coord-dual**, $\alpha = 0.5$ for **Coord-steep**, **Coord-ind**, which satisfy the condition (24), and take various values of efficiency parameter β from 0.5 to 5.0.

(i) Convergence to the optimal solution: To demonstrate our analytical findings of convergence to optimality, we first consider simple cases which support that **Coord**-algorithms find the “accurate” solution (*i.e.*, the unique NE of the game), where the exact solution can be numerically solved. Then, we show the performance of **Coord**-algorithms with two cost functions, under more general topology.

Simple cases: Let λ^* and C^* denote the optimal solution of **CG-OPT** and the maximum coordination gain of the network, respectively. We first solve the exact optimal solution at STAR-C1: $\lambda_1^* = 0.447$, $C^* = -5.218$. Parameter updates of node 1 and the total coordination gain of **Coord**-algorithms with $\beta = 5.0$ are shown in Fig. 3(a) and 3(b), respectively. We see that all three algorithms converge to the accurate solution after long iterations within a range of $O(1/\beta)$ gap, yet the convergence speeds of them do not show much difference in simple cases. Under COMP-C1, the exact optimal solution is attained at $\lambda_1^* = 0.6125$, $C^* = -5.942$, whose convergence results of node 1’s activation rate and the total coordination gain are illustrated in Fig. 3(c) and 3(d). Note that the algorithms take shorter time for convergence to the optimal solution

in `star` than in `comp` because each node has only one edge except the hub node, *i.e.*, node 1, thus pairwise interactions are less complex in `star`.

Degree of coordination at optimal solution: We provide numerical results of two types of cost functions, under a random topology. For both cases `RAND-C1` and `RAND-C2`, Fig. 4(b), 4(c) show that ***Coord-ind*** with $\alpha = 0.5$, $\beta = 4.0$ converges. Note that at the convergent status, the long-term activation rate of a node depends on the degree of coordination, *i.e.*, particularly in terms of (i) how many neighbors it has, and (ii) how powerful its neighbor is. As we see in Fig. 4(b) and 4(c), node 6 in Fig. 2(c), *i.e.*, who has very little contribution to the coordination gain since it has only one neighbor, achieves the lowest long-term activation rate, while node 13 has relatively high long-term rate. Comparing nodes 9 and 15, even though both have two neighbors, node 9 achieves a higher long-term rate since one of its neighbors (node 13) is a hub so that node 9 may implicitly contribute to coordination gain of the network via node 13. Moreover, we see that the network becomes less aggressive to be coordinated and activated if nodes have cost functions $C_i(x) = \frac{1}{1-x}$ (*i.e.*, under `RAND-C2`), since it prevents exclusive node activations.

(ii) Comparison among *Coord*-algorithms: Second, we compare the convergence of *Coord*-algorithms under `RAND-C1`. In Fig. 4(a), we observe that regarding the coordination gain, ***Coord-steep***, ***Coord-ind*** converge within 10^7 iterations, while ***Coord-dual*** still moves towards the optimal point even after 3×10^7 iterations. Note that ***Coord-dual***, ***Coord-ind*** are not designed to follow the steepest ascent direction of ***CG-OPT***, thus ***Coord-steep*** exhibits the faster convergence. Between ***Coord-dual*** and ***Coord-ind***, we expect that the rational and individual behavior when considering appropriate penalty functions in (27) brings significant improvements in the convergence rate.

(iii) Trade-off between efficiency and convergence speed: Finally, we present the numerical results that show the convergence speed and efficiency (*i.e.*, Price-of-Anarchy) of the *Coord*-algorithms for various values of β . To support that the incurring coordination gain gap due to efficiency parameter is $1/\beta$ as stated in Theorem 1 and Theorem 2, we vary β from 0.5 to 5.0 and plot the coordination gain at the converged point, and measure the convergence speed. Fig. 4(d) shows that, as β grows, *Coord*-algorithms require exponentially long time to converge, but the corresponding convergent point becomes closer to the optimal solution. From the numerical results under `RAND-C1`, coordination gain with $\beta = 4.0$ is -21.405 and converges after 2.4×10^7 iterations, while that with $\beta = 0.5$ is -22.11 and converges after 1.7×10^5 iterations.

7 Conclusion and Discussion

7.1 Summary

In many multi-agent networked environments, a variety of gains from coordinating actions of interacting agents are generated. In this paper, we first formulate an optimization problem that captures the amount of peer-to-peer coordination gain at the cost of node activation over a given network structure, and develop three distributed simulation-based algorithms relying only on one-hop message passing and local observations, which we call *Coord*-algorithms. It is inspired by a control of Ising model

in statistical physics, and theoretical findings of convergence to optimality of *Coord*-algorithms take a stochastic approximation method that runs a Markov chain incompletely over time with a smartly designed step-size function. We also provide new interpretations of *Coord-steep* and *Coord-ind* from a game-theoretic perspective.

7.2 Limitation and Future Work

In spite of theoretical findings of convergence to optimality, our *Coord*-algorithms may suffer from slow convergence for some dense graphs. Even this slow convergence issue has been observed in many prior work that use stochastic approximation theoretic update algorithms [32, 33], there also have been several efforts to expedite the convergence time [28, 36], which we believe, ensure practical values of our theoretical results. Future work includes the precise analysis of the convergence rate of *Coord*-algorithms via applying theoretical techniques, *e.g.*, with the notion of mixing time or via weak convergence theory [37, 38].

A Proof of Theorem 1

A.1 Preliminary

The convergence analysis of our *Coord*-algorithms is on the strength of stochastic approximation theory. As we will verify later, each of *Coord*-algorithms is interpreted as a stochastic approximation procedure with controlled continuous-time Markov process, where the stationary distribution of the underlying Markov process from **CDM** indeed corresponds to an Ising model. Here, we first provide preliminary results about the convergence analysis of a general stochastic approximation procedure with a controlled Markov process, where an ordinary differential equation (ODE) is usefully utilized to study the limiting behavior of the system states [28–30].

Consider a general discrete-time process $\{x[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of the following form:

$$x[t + 1] = x[t] + a[t] \cdot v(x[t], Y[t]), \quad \forall t \in \mathbb{Z}_{\geq 0}, \quad (29)$$

where $x[t] \in \mathbb{R}^L$ is L -dimensional vector representing the system state at the iteration t ; $a[t]$ corresponds to the step-size of the process; and $Y[t]$ is a random variable representing the random observation (from a Markov process) during the iteration t used to update the system state. This process is often called a *stochastic approximation with controlled continuous-time Markov process*, in [28, 29]. Here, (i) $\{z(s)\}_{s \geq 0}$ is a stochastic process taking values in a finite set \mathcal{Z} , (ii) for $s \in [t, t + 1)$, $z(s)$ evolves as a continuous-time Markov process with a control process $x[t]$, *i.e.*, with a controlled transition kernel $G^{x[t]}$, (iii) the observation $Y[t]$ is a function of $\{z(s)\}_{t \leq s < t+1}$, *i.e.*, $Y[t] = \int_t^{t+1} f(z(s))ds$, where $f(\cdot)$ is a bounded function, and (iv) $v(x, Y)$ is a bounded, continuous, Lipschitz in x and uniformly over Y . We shall assume that if $x[t] = x, \forall t$ for a fixed $x \in \mathbb{R}^L$, the controlled Markov kernel G^x is irreducible and ergodic with unique stationary distribution π^x , and furthermore, the mapping $x \mapsto G^x$ is continuous and $x \mapsto \pi^x$ is Lipschitz continuous. In the following, $\xi^x(dy)$ denotes the stationary distribution of one unit iteration, *i.e.*, $\int_0^1 f(z^x(s))ds$, where $z^x(\cdot)$ is a

Markov process with G^x , and we also assume that $x[t]$ remains bounded, which can be easily imposed by projecting the process to a bounded subset of \mathbb{R}^L . Finally, we use a positive monotonically decreasing step-size function $a[t]$ satisfying (24), where the example choices of such step-size function include $a[t] = \frac{1}{t}, \frac{1}{1+t \log t}$.

Now, define a *virtual* time-scale $\kappa(t) = \sum_{m=0}^{t-1} a[m]$. We take a continuous-time piecewise linear interpolation of the system state for the time-scale κ in the following way: define $\{x_\kappa(\tau)\}_{\tau \in \mathbb{R}_+}$ as: $\forall t \in \mathbb{Z}_{\geq 0}, \forall \tau \in [\kappa(t), \kappa(t+1))$,

$$x_\kappa(\tau) = x[t] + (x[t+1] - x[t]) \times \frac{\tau - \kappa(t)}{\kappa(t+1) - \kappa(t)}. \quad (30)$$

Intuitively, for a decreasing step-size $a[t]$, the interpolated continuous trajectory $x_\kappa(\tau)$ is an accelerated version of the original trajectory $x[t]$. Now, the following lemma provides the convergence guarantee of the iterative procedure (29).

Lemma A.1 (Theorem 1 of [30], Corollary 8 of [28](pp.74)). *Let $T > 0$, and denote by $\tilde{x}^s(\cdot)$ the solution on $[s, s+T]$ of the following ODE:*

$$\dot{x}(\tau) = \int_y v(x(\tau), y) \cdot \xi^{x(\tau)}(dy), \text{ with } \tilde{x}^s(s) = x_\kappa(s). \quad (31)$$

Then, we have almost surely,

$$\lim_{s \rightarrow \infty} \sup_{\tau \in [s, s+T]} \|x_\kappa(\tau) - \tilde{x}^s(\tau)\| = 0.$$

Note that since the Markov process is irreducible and ergodic, and f is continuous and bounded, we have,

$$\int_y v(x, y) \xi^x(dy) = \sum_{z \in \mathcal{Z}} v(x, f(z)) \pi^x(z), \quad \text{a.s.}$$

Therefore, the ODE (31) becomes the following simpler form, which will be used later in the proof of Theorem 1:

$$\dot{x}(\tau) = \sum_{z \in \mathcal{Z}} v(x(\tau), f(z)) \pi^{x(\tau)}(z). \quad (32)$$

Lemma A.1 states that as time evolves, the dynamics of the underlying Markov process is *averaged* due to the decreasing step-size, e.g., $a[t] = \frac{1}{t}$, thus “almost reaching the stationary status.” Intuitively, we expect that due to the decreasing step-size, the speed of variations of $x[t]$ decreases and tends to 0 when time sufficiently grows. As consequence, the dynamic of (29) is close to that of an irreducible and ergodic Markov process with a fixed generator (as if the system state was *frozen*), and has time to converge to its ergodic behavior. Thus, it suffices to see how the ODE (31) (equivalently (32)) behaves. Moreover, when the ODE (31) has a unique fixed stable equilibrium x^* , we have almost surely, $\lim_{t \rightarrow \infty} x[t] = x^*$.

A.2 Proof of Theorem 1 (i): Convergence

We now show the convergence of *Coord*-algorithms in Theorem 1. In particular, we prove that *Coord*-algorithms converge to the optimal solution of the approximated problem **A-CG-OPT** in (14). Our main proof strategy follows the stochastic approximation procedure whose limiting behavior is understood by an ODE as in Section A.1. For each scheme of *Coord*-algorithms, the proof contains following common two steps: first in **Step 1**, we show that the dynamics asymptotically approach some deterministic trajectory which is described as a solution trajectory of an ODE system, where each scheme tracks a slightly different deterministic trajectory. In **Step 2**, we then prove that the resulting deterministic trajectory converges to the solution of **A-CG-OPT**. To do this, we take the afore-mentioned results of Lemma A.1 into each scheme.

i) **Step 1**. In this step, we apply preliminary results in Section A.1 to each scheme of *Coord*-algorithms by showing that the original discrete sequence matches with the setup defined as (29) in Section A.1 for *Coord-dual* or an alternatively derived discrete sequence does for *Coord-steep* and *Coord-ind*. We first verify that each scheme is a stochastic approximation procedure with controlled Markov noise in (29), and then provide a lemma as a direct consequence of applying Lemma A.1 to our framework.

(a) **Coord-dual**. To follow the analysis in Section A.1, we first define a virtual time-scale $\zeta(\cdot)$ from the step-size $a[\cdot]$ of *Coord-dual* as: $\zeta(t) = \sum_{m=0}^{t-1} a[m]$. We now construct $\{\theta(\tau)\}_{\tau \in \mathbb{R}_+}$ ¹⁰, which interpolates the discrete sequence of (9) similarly to (30). We also define $\hat{s}(\tau) := \hat{s}[t] \cdot \mathbf{1}_{\zeta(t) \leq \tau < \zeta(t+1)}$, where $\mathbf{1}_A$ is the indicator function for the event A . It then should be clear that this setup matches with (29) in Section A.1. The equivalence is obtained by: $x[t] \equiv \theta[t]$; $Y[t] \equiv \hat{s}[t]$; $\{z(s)\}_{t \leq s < t+1} \equiv \{\sigma(s)\}_{t \leq s < t+1}$ is the process recording the configurations from **CDM**($\theta[t]$) during frame t ; $f(z(s)) \equiv \phi(\sigma(s))$ is a coordination configuration; $\pi^x \equiv p_\theta$ is the stationary distribution (7) of the **CDM**(θ); and finally

$$v_i(x, y) \equiv C_i'^{-1} \left(\frac{-x}{\beta} \right) - y, \quad v_{ij}(x, y) \equiv U_{ij}'^{-1} \left(\frac{x}{\beta} \right) - y.$$

Note that under our setup of utility and cost function: strictly concave, continuously twice-differentiable utility function $U_{ij} : [0, 1] \mapsto \mathbb{R}$ for edge $(i, j) \in E$ and strictly convex, continuously twice-differentiable cost function $C_i : [0, 1] \mapsto \mathbb{R}$ for node i , we have followings. First, $v_{ij}(x, y) : [\theta^{\min}, \theta^{\max}] \times [0, 1] \mapsto \mathbb{R}$ is Lipschitz continuous in x , since U_{ij} is strictly convex, continuously twice-differentiable on compact set, it follows that $U_{ij}'^{-1}$ is Lipschitz continuous by the Mean Value Theorem. Second, $v_{ij}(x, y)$ is a linear function with respect to y , thus it is obvious that it is uniformly continuous in y . Similar arguments hold for $v_i(x, y)$. Third, Markov process generated by **CDM**(θ) is a continuous function of θ , and moreover $\theta \mapsto p_\theta$ is Lipschitz continuous for the bounded $\theta \in [\theta^{\min}, \theta^{\max}]$. Therefore, one can verify that the assumptions in Section A.1 are satisfied.

(b) **Coord-steep**. Before to analyze the convergence of *Coord-steep*, we provide a detail of the derivation of the rule (16). Recall that $\mu = [\mu_\sigma]_{\sigma \in \mathcal{I}(G)}$ is the probability

¹⁰We omit ζ and use $\theta(\tau)$ instead of $\theta_\zeta(\tau)$ for notational simplicity.

distribution over the feasible configurations $\mathcal{I}(G)$, and thus the definition of $\mathcal{F}(\boldsymbol{\mu})$ in (13) can be represented in following detailed form:

$$\begin{aligned}\mathcal{F}(\boldsymbol{\mu}) &:= \sum_{(i,j) \in E} U_{ij}(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j]) - \sum_{i \in V} C_i(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i]) \\ &= \sum_{(i,j) \in E} U_{ij} \left(\sum_{\boldsymbol{\sigma} \in \mathcal{I}(G)} \mu_{\boldsymbol{\sigma}} \cdot \sigma_i \sigma_j \right) - \sum_{i \in V} C_i \left(\sum_{\boldsymbol{\sigma} \in \mathcal{I}(G)} \mu_{\boldsymbol{\sigma}} \cdot \sigma_i \right).\end{aligned}\quad (33)$$

Now, using the chain rule, a partial derivative of $\mathcal{F}(\boldsymbol{\mu})$ with respect to the variable $\mu_{\boldsymbol{\sigma}}$ is derived as follows:

$$\begin{aligned}\frac{\partial \mathcal{F}(\boldsymbol{\mu})}{\partial \mu_{\boldsymbol{\sigma}}} &= \sum_{(i,j) \in E} U'_{ij}(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j]) \cdot \frac{\partial \mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j]}{\partial \mu_{\boldsymbol{\sigma}}} - \sum_{i \in V} C'_i(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i]) \cdot \frac{\partial \mathbb{E}_{\boldsymbol{\mu}}[\sigma_i]}{\partial \mu_{\boldsymbol{\sigma}}} \\ &= \sum_{(i,j) \in E} U'_{ij}(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j]) \cdot \sigma_i \sigma_j - \sum_{i \in V} C'_i(\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i]) \cdot \sigma_i.\end{aligned}\quad (34)$$

Now, the first step is to approximate **Coord-steep** for large t by the dynamic of a continuous-time ODE system, by taking a continuous-time interpolation. While a stochastic approximation idea in **Coord-dual** comes from the diminishing step-size $a[\cdot]$, adopting the cumulative rate $\bar{s}[t]$ in **Coord-steep** plays the similar role (see the relation (19)). To understand the role of $\bar{s}[t]$, we introduce an *alternative* discrete-time sequence $\{\boldsymbol{\rho}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ derived from $\{\boldsymbol{\theta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of **Coord-steep** in (10) defined as:

$$\boldsymbol{\rho}[t] = \frac{1}{\alpha} \cdot \boldsymbol{\theta}[t] + \left(1 - \frac{1}{\alpha}\right) \cdot \boldsymbol{\theta}[t-1],$$

and thus we have the following property by applying recursion:

$$\begin{aligned}\boldsymbol{\theta}[t] &= \alpha \cdot \boldsymbol{\rho}[t] + (1 - \alpha) \cdot \boldsymbol{\theta}[t-1] \\ &= \alpha \boldsymbol{\rho}[t] + (1 - \alpha) \left(\alpha \boldsymbol{\rho}[t-1] + (1 - \alpha) \boldsymbol{\theta}[t-2] \right) \\ &= \dots = \sum_{m=0}^{t-1} \alpha (1 - \alpha)^m \boldsymbol{\rho}[t-m].\end{aligned}\quad (35)$$

Then, from (10) and (35), **Coord-steep** can be understood as the update rule $\{\boldsymbol{\rho}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of following form:

$$\rho_i[t+1] = -\beta C'_i(\bar{s}_i[t]), \quad \rho_{ij}[t+1] = \beta U'_{ij}(\bar{s}_{ij}[t]),\quad (36)$$

and thus we have

$$\bar{s}_i[t-1] = C_i'^{-1} \left(\frac{-\rho_i[t]}{\beta} \right), \quad \bar{s}_{ij}[t-1] = U_{ij}'^{-1} \left(\frac{\rho_{ij}[t]}{\beta} \right).$$

Now, when t grows large, the update rule (36) becomes approximately as follows under the assumption **(A1)**,

$$\rho_i[t+1] = -\beta C'_i(\bar{s}_i[t])$$

$$\begin{aligned}
&\stackrel{(a)}{=} -\beta \left(C'_i(\bar{s}_i[t-1]) + \frac{1}{t} (\hat{s}_i[t] - \bar{s}_i[t-1]) C''_i(\bar{s}_i[t-1]) \right) \\
&= -\beta C'_i(\bar{s}_i[t-1]) + \frac{1}{t} g_i(\rho_i[t]) (\bar{s}_i[t-1] - \hat{s}_i[t]) \\
&\stackrel{(b)}{=} \rho_i[t] + \frac{1}{t} g_i(\rho_i[t]) \left(C_i'^{-1} \left(\frac{-\rho_i[t]}{\beta} \right) - \hat{s}_i[t] \right), \tag{37}
\end{aligned}$$

and similarly,

$$\rho_{ij}[t+1] = \rho_{ij}[t] + \frac{1}{t} g_{ij}(\rho_{ij}[t]) \left(U_{ij}'^{-1} \left(\frac{\rho_{ij}[t]}{\beta} \right) - \hat{s}_{ij}[t] \right),$$

where $g_i(\cdot)$, $g_{ij}(\cdot)$ are defined in (23), and both are positive for convex, increasing function $C_i(\cdot)$ and concave, increasing function $U_{ij}(\cdot)$. The equality (a) holds from a first-order Taylor's expansion, (b) comes from the **Coord-steep** rule.

To take afore-mentioned results in Section A.1 into this framework, we define another virtual time-scale $\kappa(t)$ as $\kappa(t) := \sum_{m=1}^{t-1} \frac{1}{m}$ with $\kappa(0) = 0$, since the discrete sequence of **Coord-steep** is interpreted to have a step-size $\frac{1}{t}$ at iteration t , see (37). We construct an interpolated trajectory $\{\rho(\tau)\}_{\tau \in \mathbb{R}_+}$ from the discrete sequence $\{\rho[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ in (36) with time-scale κ . Then, it should be clear that the alternative process matches with the setup (29) in Section A.1. The equivalence is obtained by $x[t] \equiv \rho[t]$; $Y[t] \equiv \hat{s}[t]$; $a[t] = \frac{1}{t}$; $\{z(s)\}_{t \leq s < t+1} \equiv \{\sigma(s)\}_{t \leq s < t+1}$; $f(z(s)) \equiv \phi(\sigma(s))$; $\pi^x \equiv p_\theta$ in (7); and finally

$$\begin{aligned}
v_i(x, y) &\equiv g_i(x) \left(C_i'^{-1} \left(\frac{-x}{\beta} \right) - y \right), \quad i \in V, \\
v_{ij}(x, y) &\equiv g_{ij}(x) \left(U_{ij}'^{-1} \left(\frac{x}{\beta} \right) - y \right), \quad (i, j) \in E.
\end{aligned}$$

(c) **Coord-ind.** Here, we also introduce an alternative discrete-time sequence $\{\eta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ derived from $\{\theta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of **Coord-ind** in (11) as follows:

$$\eta[t] = \frac{1}{\gamma[t-1]} \cdot \theta[t] + \left(1 - \frac{1}{\gamma[t-1]} \right) \cdot \theta[t-1],$$

where $\gamma[t]$ is given by

$$\gamma_i[t] = \frac{\alpha}{\beta} \frac{\partial s_i(\theta[t])}{\partial \theta_i}, \quad \gamma_{ij}[t] = \frac{\alpha}{\beta} \frac{\partial s_{ij}(\theta[t])}{\partial \theta_{ij}}. \tag{38}$$

Then, we have the following property by applying recursion:

$$\begin{aligned}
\theta[t] &= \gamma[t-1] \cdot \eta[t] + (1 - \gamma[t-1]) \cdot \theta[t-1] \\
&= \gamma[t-1] \eta[t] \\
&\quad + \sum_{m=1}^{t-1} \prod_{l=1}^m \left(1 - \gamma[t-l] \right) \gamma[t-m-1] \eta[t-m]. \tag{39}
\end{aligned}$$

Now, from (11) and (39), **Coord-ind** can be understood as the update rule $\{\boldsymbol{\eta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ of representation in (22). Then, two sequences $\{\boldsymbol{\rho}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ in (36) and $\{\boldsymbol{\eta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ in (22) are evolved in the same way. Therefore, an interpolated trajectory $\{\boldsymbol{\eta}(\tau)\}_{\tau \in \mathbb{R}_+}$ from $\{\boldsymbol{\eta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ with time-scale κ is equivalent to $\{\boldsymbol{\rho}(\tau)\}_{\tau \in \mathbb{R}_+}$.

From the equivalence above, following is a direct consequence from Lemma A.1 in Section A.1, which states that the interpolated trajectory of each scheme asymptotically tracks the solution trajectory of the corresponding ODE system.

Lemma A.2. *Let $T > 0$, and fix $w > 0$. (i) **Coord-dual.** Denote by $\tilde{\boldsymbol{\theta}}^w(\cdot)$ the solution on $[w, w + T]$ of the following ODE: $\forall i \in V$ and $\forall (i, j) \in E$,*

$$\begin{aligned}\dot{\theta}_i(\tau) &= C_i^{\prime-1}\left(\frac{-\theta_i(\tau)}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}(\tau)}[\sigma_i], \\ \dot{\theta}_{ij}(\tau) &= U_{ij}^{\prime-1}\left(\frac{\theta_{ij}(\tau)}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}(\tau)}[\sigma_i \sigma_j],\end{aligned}\quad (40)$$

with $\tilde{\boldsymbol{\theta}}^w(w) = \boldsymbol{\theta}(w)$. Then, we have almost surely,

$$\lim_{w \rightarrow \infty} \sup_{\tau \in [w, w+T]} \|\boldsymbol{\theta}(\tau) - \tilde{\boldsymbol{\theta}}^w(\tau)\| = 0.$$

(ii) **Coord-steep and Coord-ind.** Denote by $\tilde{\boldsymbol{\rho}}^w(\cdot)$ the solution on $[w, w + T]$ of the following ODE: $\forall i \in V$ and $\forall (i, j) \in E$,

$$\begin{aligned}\dot{\rho}_i(\tau) &= g_i(\rho_i(\tau)) \left[C_i^{\prime-1}\left(\frac{-\rho_i(\tau)}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}(\tau)}[\sigma_i] \right], \\ \dot{\rho}_{ij}(\tau) &= g_{ij}(\rho_{ij}(\tau)) \left[U_{ij}^{\prime-1}\left(\frac{\rho_{ij}(\tau)}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}(\tau)}[\sigma_i \sigma_j] \right],\end{aligned}\quad (41)$$

with $\tilde{\boldsymbol{\rho}}^w(w) = \boldsymbol{\rho}(w) = \boldsymbol{\eta}(w)$. Then, we have almost surely,

$$\lim_{w \rightarrow \infty} \sup_{\tau \in [w, w+T]} \|\boldsymbol{\rho}(\tau) - \tilde{\boldsymbol{\rho}}^w(\tau)\| = 0,$$

and equivalently,

$$\lim_{w \rightarrow \infty} \sup_{\tau \in [w, w+T]} \|\boldsymbol{\eta}(\tau) - \tilde{\boldsymbol{\rho}}^w(\tau)\| = 0.$$

ii) Step 2. Now, we prove that each ODE system in Lemma A.2 has a unique fixed point, and thus the resulting deterministic solution trajectory converges to the point. We then show that the point attains at the optimal solution of **A-CG-OPT**, i.e., **Coord**-algorithms converge to $\boldsymbol{\theta}^\circ$.

(a) **Coord-dual.** We show that the ODE system (40) has the solution of **A-CG-OPT**, denoted by $\boldsymbol{\theta}^\circ$, as a unique fixed point. In particular, (40) may be interpreted as a sub-gradient dynamics solving the *dual* of the convex problem **A-CG-OPT**. To that end, we first consider the Lagrangian \mathcal{L} of **A-CG-OPT** with dual variables $\mathbf{k} = ([k_i]_{i \in V}, [k_{ij}]_{(i,j) \in E})$:

$$\mathcal{L}(\boldsymbol{\mu}, \boldsymbol{\lambda}; \mathbf{k}) = \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}) - \sum_{i \in V} C_i(\lambda_i) + \frac{1}{\beta} H(\boldsymbol{\mu})$$

$$+ \sum_{i \in V} k_i (\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i] - \lambda_i) + \sum_{(i,j) \in E} k_{ij} (\mathbb{E}_{\boldsymbol{\mu}}[\sigma_i \sigma_j] - \lambda_{ij}). \quad (42)$$

The primal solution of **A-CG-OPT** is the minimum point of the dual function, which is given by $\mathcal{D}(\mathbf{k}) = \sup_{\boldsymbol{\mu}, \boldsymbol{\lambda}} \mathcal{L}(\boldsymbol{\mu}, \boldsymbol{\lambda}; \mathbf{k})$. Finally, the dual problem is formulated as:

$$\min_{\mathbf{k} \in \mathbb{R}^{|\mathcal{V}|+|\mathcal{E}|}} \mathcal{D}(\mathbf{k}). \quad (43)$$

Note that the primal problem in (14) is a concave maximization and the dual problem in (43) is a convex minimization from the concavity of entropy and $U_{ij}(\cdot)$, convexity of $C_i(\cdot)$ under our setup. Therefore, following the results in standard primal-dual optimization theory, there is no duality gap and both have the same, unique solution, and moreover its sub-gradient algorithm will converge to the solution.

Given a feasible dual variable \mathbf{k} , let $(\boldsymbol{\mu}^\dagger(\mathbf{k}), \boldsymbol{\lambda}^\dagger(\mathbf{k}))$ be the corresponding feasible primal solution that maximizes the Lagrangian \mathcal{L} . Given the structure of \mathcal{L} in (42), from Karush-Kuhn-Tucker (KKT) conditions of **A-CG-OPT**, it follows that $\boldsymbol{\mu}^\dagger(\mathbf{k}), \boldsymbol{\lambda}^\dagger(\mathbf{k})$ should be such that:

$$\begin{aligned} \boldsymbol{\mu}_\sigma^\dagger(\mathbf{k}) &\propto \exp\left(\sum_{i \in V} \beta k_i \sigma_i + \sum_{(i,j) \in E} \beta k_{ij} \sigma_i \sigma_j\right), \quad \forall \sigma \in \mathcal{I}(G), \\ \lambda_i^\dagger(\mathbf{k}) &= \arg \max_{y \in [0,1]} \left[-C_i(y) - k_i y\right], \quad \forall i \in V, \\ \lambda_{ij}^\dagger(\mathbf{k}) &= \arg \max_{y \in [0,1]} \left[U_{ij}(y) - k_{ij} y\right], \quad \forall (i,j) \in E. \end{aligned} \quad (44)$$

Now, we can conclude that $\boldsymbol{\mu}^\dagger(\boldsymbol{\theta}) = p_{\boldsymbol{\theta}}$ with $\boldsymbol{\theta} = \beta \mathbf{k}$, from (7) and (44). Then, the dual function is represented with respect to $\boldsymbol{\theta}$ as $\mathcal{D}(\boldsymbol{\theta}) = \mathcal{L}(\boldsymbol{\mu}^\dagger(\boldsymbol{\theta}), \boldsymbol{\lambda}^\dagger(\boldsymbol{\theta}); \boldsymbol{\theta})$, and the slack in each constraint is given by:

$$\mathbb{E}_{\boldsymbol{\mu}^\dagger(\boldsymbol{\theta})}[\sigma_i] - \lambda_i^\dagger(\boldsymbol{\theta}), \quad \text{and} \quad \mathbb{E}_{\boldsymbol{\mu}^\dagger(\boldsymbol{\theta})}[\sigma_i \sigma_j] - \lambda_{ij}^\dagger(\boldsymbol{\theta}). \quad (45)$$

Accounting for (44) and (45), the sub-gradient algorithm solving the dual problem (43) with parameter $\boldsymbol{\theta}$, *i.e.*, using $\nabla \mathcal{D}(\boldsymbol{\theta})$, is given by following ODEs: $\forall i \in V, \forall (i,j) \in E$,

$$\dot{\theta}_i = C_i'^{-1}\left(\frac{-\theta_i}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}}[\sigma_i], \quad \dot{\theta}_{ij} = U_{ij}'^{-1}\left(\frac{\theta_{ij}}{\beta}\right) - \mathbb{E}_{\boldsymbol{\theta}}[\sigma_i \sigma_j]. \quad (46)$$

which is obviously equivalent to (40), provided that $\boldsymbol{\theta}(\tau)$ remains between $[\theta^{\min}, \theta^{\max}]$ component-wisely. Note that the dual solution, denoted by $\boldsymbol{\theta}^\circ$, actually belongs to the interval $[\theta^{\min}, \theta^{\max}]$ component-wisely, as a fixed point of (46), under **(A1)**. Therefore, the sub-gradient converges to the dual solution $\boldsymbol{\theta}^\circ$, where the corresponding primal solution is $(\boldsymbol{\mu}^\circ, \boldsymbol{\lambda}^\circ)$ ¹¹, and hence the solution trajectory of the ODE system (40) also does. Finally, we can conclude that under **Coord-dual**, we have almost surely, $\lim_{t \rightarrow \infty} \boldsymbol{\theta}[t] = \boldsymbol{\theta}^\circ$.

(b) Coord-steep. In case of **Coord-steep**, we need an additional step that proves the equivalence of the convergence of alternative process $\{\boldsymbol{\rho}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ and that of $\{\boldsymbol{\theta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$.

¹¹It is obvious that we mean $\boldsymbol{\mu}^\circ = \boldsymbol{\mu}^\dagger(\boldsymbol{\theta}^\circ) = p_{\boldsymbol{\theta}^\circ}$ and $\boldsymbol{\lambda}^\circ = \boldsymbol{\lambda}^\dagger(\boldsymbol{\theta}^\circ)$.

Since parameter lies in compact region, from (35), there exist constants L and M such that $\forall t \in \mathbb{Z}_{\geq 0}$,

$$|\rho_i[t]| < L, \quad \left| g_i(\rho_i[t]) \left(C_i'^{-1} \left(\frac{-\rho_i[t]}{\beta} \right) - \hat{s}_i[t] \right) \right| < M. \quad (47)$$

For $\epsilon > 0$, let $T(\epsilon) := \frac{4 \log(\frac{\epsilon M}{4L})}{\epsilon \log(1-\alpha)}$. Then, for all $t \geq T(\epsilon)$, $|\rho_i[t] - \theta_i[t]| \leq \frac{5}{4}\epsilon M$, because¹²

$$\begin{aligned} |\rho_i[t] - \theta_i[t]| &\stackrel{(a)}{=} \left| \rho_i[t] - \sum_{m=0}^{t-1} \rho_i[t-m] \alpha (1-\alpha)^m \right| \\ &\leq \sum_{m=0}^{t-1} |\rho_i[t] - \rho_i[t-m]| \alpha (1-\alpha)^m + |\rho_i[t]| (1-\alpha)^t \\ &\stackrel{(b)}{\leq} \frac{\epsilon t/4}{t - \epsilon t/4} M + 2L \sum_{m=\epsilon t/4}^{t-1} \alpha (1-\alpha)^m + L(1-\alpha)^{\epsilon t/4} \\ &\stackrel{(c)}{\leq} \frac{\epsilon}{2} M + 2L(1-\alpha)^{\epsilon t/4} + L(1-\alpha)^{\epsilon t/4} \stackrel{(d)}{\leq} \frac{5}{4}\epsilon M, \end{aligned}$$

where (a) comes from (35), (b) and (c) come from the following using triangle inequality and (37), (47):

$$\begin{aligned} &\sum_{m=0}^{\epsilon t/4-1} |\rho_i[t] - \rho_i[t-m]| \alpha (1-\alpha)^m \\ &\leq \sum_{m=1}^{\epsilon t/4-1} \sum_{k=1}^m |\rho_i[t-k+1] - \rho_i[t-k]| \alpha (1-\alpha)^m \\ &\leq \sum_{m=1}^{\epsilon t/4-1} \frac{m \cdot M}{t-m} \alpha (1-\alpha)^m \leq \frac{\epsilon t/4}{t - \epsilon t/4} M, \end{aligned}$$

and finally (d) holds for $t \geq T(\epsilon)$ and $\epsilon \leq 2$. Therefore, we have the following relation between $\rho[t]$ and $\theta[t]$:

$$\lim_{t \rightarrow \infty} \rho[t] - \theta[t] = 0. \quad (48)$$

Now, we observe that the ODE system (41) is equivalent to (40), because we have positive $g_i(\cdot)$, $g_{ij}(\cdot)$. Therefore, the ODE system (41) converges to a unique fixed point, say ρ° , such that $\theta^\circ = \rho^\circ$. Finally, from (48), we can conclude that under **Coord-steep**, we have almost surely, $\lim_{t \rightarrow \infty} \theta[t] = \theta^\circ$.

(c) **Coord-ind**. In case of **Coord-ind**, similarly to the earlier scheme **Coord-steep**, we prove the equivalence of the convergence of $\{\eta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ and that of $\{\theta[t]\}_{t \in \mathbb{Z}_{\geq 0}}$. From the simple algebra of (8), we have component-wisely:

$$\nabla \mathbf{s}(\theta) = \mathbf{s}(\theta) \left(1 - \mathbf{s}(\theta) \right). \quad (49)$$

¹²Here, we use just $\epsilon t/4$ instead of $\lceil \epsilon t/4 \rceil$ for notional simplicity.

We first denote by γ^{\min} and γ^{\max} the minimum and maximum value of the sequence $\{\gamma[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ in (38), respectively. Note that for sufficiently large β , $\gamma^{\min}, \gamma^{\max}$ is less than 1, since $\nabla s(\boldsymbol{\theta}) \in (0, 1/4)$ from (49). Now, since parameter $\boldsymbol{\theta}$ lies in compact region, from (39), there also exist constants L and M such that $\forall t \in \mathbb{Z}_{\geq 0}$,

$$|\eta_i[t]| < L, \quad \text{and} \quad \left| g_i(\eta_i[t]) \left(C_i'^{-1} \left(\frac{-\eta_i[t]}{\beta} \right) - \hat{s}_i[t] \right) \right| < M.$$

For $\epsilon > 0$, let $S(\epsilon) := \frac{4 \log(\min(\frac{\epsilon M}{4L}, (\frac{\epsilon M}{4L} - \frac{\gamma^{\min}}{\gamma^{\max}}) \cdot (\frac{\gamma^{\max}}{\gamma^{\min}})^2))}{\epsilon \log(1 - \gamma^{\min})}$. Then, for all $t \geq S(\epsilon)$ and $\epsilon \leq 2$, similar argument in *Coord-steep* can be done to show $|\eta_i[t] - \theta_i[t]| \leq \frac{5}{4} \frac{\gamma^{\max}}{\gamma^{\min}} \epsilon M$. Now, we have the following relation between $\boldsymbol{\eta}[t]$ and $\boldsymbol{\theta}[t]$ that:

$$\lim_{t \rightarrow \infty} \boldsymbol{\eta}[t] - \boldsymbol{\theta}[t] = 0. \quad (50)$$

From the equivalence of $\{\boldsymbol{\rho}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$ and $\{\boldsymbol{\eta}[t]\}_{t \in \mathbb{Z}_{\geq 0}}$, combined with Lemma A.2 and the argument of the ODE system (41) in *Coord-steep*, we can conclude that under *Coord-ind*, we have almost surely, $\lim_{t \rightarrow \infty} \boldsymbol{\theta}[t] = \boldsymbol{\theta}^\circ$.

Consequently, from *Step 1* and *Step 2*, we complete the proof that under all *Coord*-algorithms, $\boldsymbol{\theta}[t]$ converges to $\boldsymbol{\theta}^\circ$, where the optimal solution of **A-CG-OPT** is attained. \square

A.3 Proof of Theorem 1 (ii): Optimality

We now show the asymptotic optimality of *Coord*-algorithms by verifying the relation between the solution of **A-CG-OPT** and that of **CG-OPT**. In Section A.2, we have shown that the result of *Coord*-algorithms converges to $\boldsymbol{\theta}^\circ$ such that the corresponding $(p_{\boldsymbol{\theta}^\circ}, \mathbf{s}(\boldsymbol{\theta}^\circ))$ is the solution of **A-CG-OPT** in (14). To establish a goodness of the result of *Coord*-algorithms, note that the optimal solution of **CG-OPT** in (3), denoted by $\boldsymbol{\lambda}^*$, along with an appropriate distribution $\boldsymbol{\mu}^* \in \mathcal{M}$, is one feasible solution of the problem **A-CG-OPT**. Therefore, it follows that

$$\begin{aligned} & \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^*) - \sum_{i \in V} C_i(\lambda_i^*) \\ & \stackrel{(a)}{\leq} \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^*) - \sum_{i \in V} C_i(\lambda_i^*) + \frac{1}{\beta} H(\boldsymbol{\mu}^*) \\ & \stackrel{(b)}{\leq} \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^\circ) - \sum_{i \in V} C_i(\lambda_i^\circ) + \frac{1}{\beta} H(\boldsymbol{\mu}^\circ) \\ & \stackrel{(c)}{\leq} \sum_{(i,j) \in E} U_{ij}(\lambda_{ij}^\circ) - \sum_{i \in V} C_i(\lambda_i^\circ) + \frac{\log |\mathcal{I}(G)|}{\beta}. \end{aligned}$$

In the above, the first inequality (a) comes from the fact that the entropy is non-negative, (b) holds since $(\boldsymbol{\mu}^\circ, \boldsymbol{\lambda}^\circ)$ is the optimal solution of **A-CG-OPT**, and finally we have used the fact that the maximum value of a discrete-valued random variable's entropy is at most the logarithm of the cardinality of the support set $|\mathcal{I}(G)|$, for the last inequality (c). \square

B Proof of Theorem 2

Proof. (i) Uniqueness. We prove the existence and the uniqueness of non-trivial NE in our game $\text{CoordGain}(\beta)$ using a potential game approach. Consider the following function $P(\boldsymbol{\theta})$ on the space $\mathcal{C}^+ = \{\boldsymbol{\theta} : s(\boldsymbol{\theta}) > 0\}$, *i.e.*, the set of strategies producing “non-trivial” rates, defined by:

$$P(\boldsymbol{\theta}) := - \sup_{\boldsymbol{\mu} \in \mathcal{M}, \boldsymbol{\lambda} \in [0,1]^{|V|+|E|}} \mathcal{L}(\boldsymbol{\mu}, \boldsymbol{\lambda}; \frac{\boldsymbol{\theta}}{\beta}), \quad (51)$$

where $\mathcal{L}(\cdot)$ is defined in (42) to describe a dual function of **A-CG-OPT**. It is easy to check that $P(\boldsymbol{\theta})$ is strictly concave in $\boldsymbol{\theta}$ since $P(\cdot)$ is the supremum of $\mathcal{L}(\cdot)$, which is a family of affine functions in $\boldsymbol{\theta}$. We now show that $\text{CoordGain}(\beta)$ for any constant $\beta > 0$ is an *ordinal potential game* [39] with the potential function $P(\boldsymbol{\theta})$, *i.e.*, $\text{sgn} \frac{\partial \Psi_n(\boldsymbol{\theta})}{\partial \theta_n} = \text{sgn} \frac{\partial P(\boldsymbol{\theta})}{\partial \theta_n}$, for all players $n \in N$. To verify this, we first have derivative of each player’s payoff function: for each node player $i \in V$,

$$\begin{aligned} \frac{\partial \Psi_i(\boldsymbol{\theta})}{\partial \theta_i} &= - \frac{\partial}{\partial \theta_i} \left(C_i(s_i(\boldsymbol{\theta})) + \frac{1}{\beta} \int_{-\infty}^{\theta_i} x s'_i(x, \theta_{-i}) dx \right) \\ &= - \frac{\partial s_i(\boldsymbol{\theta})}{\partial \theta_i} \left(C'_i(s_i(\boldsymbol{\theta})) + \frac{\theta_i}{\beta} \right) \\ &= -s_i(\boldsymbol{\theta}) \left(1 - s_i(\boldsymbol{\theta}) \right) \left(C'_i(s_i(\boldsymbol{\theta})) + \frac{\theta_i}{\beta} \right), \end{aligned}$$

where the last equality comes from (49). Similarly, we also have for each edge player $(i, j) \in E$,

$$\frac{\partial \Psi_{ij}(\boldsymbol{\theta})}{\partial \theta_{ij}} = s_{ij}(\boldsymbol{\theta}) \left(1 - s_{ij}(\boldsymbol{\theta}) \right) \left(U'_{ij}(s_{ij}(\boldsymbol{\theta})) - \frac{\theta_{ij}}{\beta} \right).$$

Second, we have derivative of a potential function as:

$$\frac{\partial P(\boldsymbol{\theta})}{\partial \theta_i} = C'_i{}^{-1} \left(\frac{-\theta_i}{\beta} \right) - s_i(\boldsymbol{\theta}), \quad \text{and} \quad \frac{\partial P(\boldsymbol{\theta})}{\partial \theta_{ij}} = U'_{ij}{}^{-1} \left(\frac{\theta_{ij}}{\beta} \right) - s_{ij}(\boldsymbol{\theta}).$$

Therefore, on the space \mathcal{C}^+ , $\text{sgn} \frac{\partial \Psi_n(\boldsymbol{\theta})}{\partial \theta_n} = \text{sgn} \frac{\partial P(\boldsymbol{\theta})}{\partial \theta_n}$ for each player $n \in N$. From the standard results in potential game theory and strict concavity of $P(\cdot)$, the solution that maximizes $P(\cdot)$ is a NE $\boldsymbol{\theta}^{\text{NE}}$, where each player’s strategy is a best response to the others’ strategies, *i.e.*, $\nabla \Psi_n(\boldsymbol{\theta}) = 0, \forall n \in N$, and moreover it is unique and non-trivial.

(ii) Price-of-Anarchy. The proof of Price-of-Anarchy follows the same argument in Section A.3, since we observe that the unique non-trivial NE in our game $\boldsymbol{\theta}^{\text{NE}}$ coincides with the optimal solution of **A-CG-OPT**, *i.e.*, $\boldsymbol{\theta}^{\text{NE}} = \boldsymbol{\theta}^\circ$. From the analysis in Section A.3, we can easily verify that

$$\text{PoA} = \frac{\sum_{(i,j) \in E} U_{ij}(s_{ij}(\boldsymbol{\theta}^*)) - \sum_{i \in V} C_i(s_i(\boldsymbol{\theta}^*))}{\sum_{(i,j) \in E} U_{ij}(s_{ij}(\boldsymbol{\theta}^{\text{NE}})) - \sum_{i \in V} C_i(s_i(\boldsymbol{\theta}^{\text{NE}}))}$$

$$= 1 + O\left(\frac{\log |\mathcal{I}(G)|}{\beta}\right),$$

and thus we have $\lim_{\beta \rightarrow \infty} \text{PoA} = 1$. □

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