Short Queue Behavior and Rate Based Marking

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***Abstract*— Differential equation models for Internet congestion control algorithms have been widely used to understand network dynamics and the design of router algorithms. These models use a fluid approximation for user data traffic, and describe the dynamics of the router queue and user adaptation through coupled differential equations.**

In this paper, we show that the randomness due to short and unresponsive flows in the Internet is sufficient to decouple the dynamics of the router queues from those of the end controllers. We show that this implies that a time-scale decomposition naturally occurs such that the dynamics of the router manifest only through their *statistical steady-state* behavior.

The interaction between the routers and flows occur through marking, where routers indicate congestion by appropriately marking packets during congestion. In this paper, we show that the time-scale decomposition implies that a queue-length based marking function such as Random Early Detection (RED) or Random Exponential Marking (REM) have an *equivalent* form which depend *only on the data arrival rate from the end-systems and do not depend on the queue dynamics*. This leads to much simpler dynamics of the differential equation models (there is no queueing dynamics to consider), which enables easier simulation (the state space is reduced) and analysis. We finally validate our analysis through simulation.

Keywords: Internet congestion control, time-scale decomposition, marking functions

I. Introduction

We consider the problem of Internet congestion control when the network is accessed by a mixture of long-lived controlled flows, as well as short-flows which do not react to congestion. The short flows model a mixture of real-time based traffic (such as real-time multimedia) as well as web traffic (so called web-mice), where the sessions are too short for the end systems to react to network congestion.

The transmission rate of the long-lived flows are controlled by the intermediate routers in the network. The task of these routers is to simply notify the end systems whenever they detect congestion in the network. Associated with each router is a marking function, which *marks* a fraction of the flow, and the fraction that is marked is a function of the arrival rate (rate based marking) or the queue length (queue based marking). In the Internet, marking is implemented via the Explicit Congestion Notification (ECN) mechanism [1], where packets have a bit in the header that can be set to '1' to indicate congestion. The end-host reacts to this information by suitably adapting it's transmission rate, thus adapting to network congestion.

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There has been extensive research on differential equation based congestion control [2], [3], [4], [5], [6], [7], where fluid models of a large number of flows were used to model the dynamics of the system based on a rate based marking scheme. The source controllers are modeled by differential equations, which inject a fluid into the network. These controllers adapt their transmission rate based on network feedback in the form of a fraction of fluid that is marked by the routers. In other words, with n flows in the network, the dynamics of the controller are described by

$$\dot{x}_n^i(t) \quad = \quad \kappa\left(w - x_n^i(t)p_r\left(\frac{1}{n}\sum_{j=1}^{n}(a_n^j(t)+x_n^j(t))\right)\right) \tag{1}$$

where w, κ are parameters of the controller that determine the equilibrium rate as well as the transient dynamics. $x_n^i(t)$ is the transmission rate of the controlled flow i at time t, and $\sum_i a_n^i(t)$ represents the short-lived uncontrolled flows. The function $p_r(\cdot)$ is a *rate based marking* function whose argument is *the average arrival rate to the router* (additional discussion is available later in this section). The marking function indicates the level of congestion at the router. Thus, $p_r(\cdot)$ is a monotone, increasing function with range $[0,1]$. The larger the marking level is, the higher is the perceived congestion at the router. As seen (1), the controller reacts to a congestion level by decreasing the transmission rate.

Alternately, instead of adapting based on the average arrival rate, the marking function at the router can adapt based on the *queue length at the router.* In other words, the router is associated with a *queue based marking* function $p_q(\cdot)$. This is assumed to be a monotone increasing function over $[0,1]$, and Lipschitz continuous with parameter L_q. The associated differential equation model for the end-system controller is given by

$$\begin{aligned}\dot{x}_n^i(t) &= \kappa\left(w - x_n^i(t)p_q\left(\frac{1}{n}Q_n(t)\right)\right)\\ \dot{Q}_n(t) &= \begin{cases}\sum_{j=1}^{n}(a_n^j(t)+x_n^j(t)) - nc & \text{if } \mathrm{Q^n(t)} > 0,\\ \left[\sum_{j=1}^{n}(a_n^j(t)+x_n^j(t)) - nc\right]^+ & \text{if } \mathrm{Q^n(t)} = 0,\end{cases}\end{aligned} \tag{2}$$

where $Q_n(t)$ is the queue length at the router, and nc is the capacity of the link. Examples of queue based marking include Random Early Discard (RED) [1], Adaptive Virtual Queue (AVQ) [8], and Random Exponential Marking (REM) [9].

It has been shown in [6] that the differential equation based models described in (1) and (2) are valid models of in the Internet when there are large enough number of flows and the network capacity is large (scaled with the number of flows). In such a regime, the arguments of the marking functions are interpreted as the *average arrival rate* (averaging by the number of flows) or the *scaled queue length* (scaled by the number of flows) respectively. Physically, this scaling of the arguments correspond to the fact that the arrival rates and capacity are large, see [6] for details.

In other words, for a network model with n flows and the capacity at the router being nc, the marking function at the router adapts either based on the average arrival rate $x(t) = \frac{1}{n}X^n(t)$, where $X^n(t)$ is the total arrival rate to the router, or based on the average queue length $q(t) = \frac{1}{n}Q^n(t)$, where $Q^n(t)$ is the queue length at the router. In particular, this implies that *as the system size becomes larger, so does the associated queue length at the router*. In other words, a finite non-zero queue length in the fluid differential equation model (2) indicates that the *actual* queue length in the router is large (of order n).

However, as link speeds in modern and future communication networks is becoming higher, high-speed memory buffer with high cost is required in the design of such networks. Therefore, it is controversial that queue buffer at intermediate routers *scale linearly* with the number of flows [10]. In [10], [11], the authors have shown that buffers need not scale with the link speed in order to achieve significant multiplexing gains.

In this paper, we focus on this regime where the queue length *does not scale with the number of flows*. Such a behavior occurs, for instance, if the queue based marking function $p_q(\cdot)$ is invariant with the number of flows and *is a function of the actual queue length, not the average queue length*. Under such a regime, the queue dynamics occur on a much *faster* time-scale than that of the end system controller [12]. In this context, it is reasonable to expect that queueing dynamics are not visible to the end system controller. Instead, the queueing behavior at the router affects the end system controller only through the statistical behavior of the queue.

This paper quantifies the above heuristic by showing that under suitable assumptions, *the queue based marking and the associated queueing dynamics can be approximated by a rate based marking function given by*

$$p(x) \triangleq E_{\pi^x}[p_q(Q)],$$

where π^x is the stationary queue-length distribution of an M/D/1 queue with Poisson arrival rate λ and capacity $(c-x)$. The parameter x is simply the average arrival rate from the controlled flows (averaging over flows, not time) to the router queue.

This implies that a Internet differential equation model can be composed solely of rate based controllers, even if queue-based marking is employed. This leads to much simpler dynamics of the differential equation model (there is no queueing dynamics to consider), which enables easier simulation (the state space is reduced) and analysis.

II. System Model

Consider the system shown in Figure 1. We consider a single queue with the FIFO (First In First Out) scheduling discipline accessed by two types of flows: *(i) controlled flows and (ii) uncontrolled flows.* We consider a sequence of systems indexed by n, the scaling parameter. In the n-th system, the queue is fed by n independent identically distributed uncontrolled flows and by n controlled flows determined by a congestion control algorithm. The output capacity of the router queue scaled with n as nc packets per second.

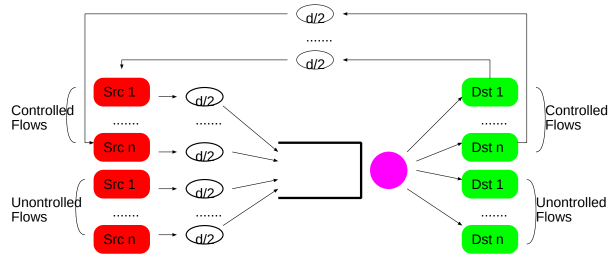

Fig. 1. System Model

For the n-th system, we model each uncontrolled flow by means of a point process $A_i^n(t)$, that represents the cumulative number of packets from flow i that arrive until time t. We assume that each $A_n^i(t)$ has the same distribution as a simple stationary point process A that satisfies the following assumptions [10], [13].

Assumption 2.1: A is a simple stationary point process satisfying the following three properties.

(i) There exists $\lambda > 0$ such that $E[A(t)] = \lambda t$ for $t \in [0, \infty)$.

(ii) There exists $\theta_0 > 0$ and $K < \infty$ such that

$$\lim_{t \to 0^+} E[e^{\theta_0 A_t} \mathbf{1}_{A(t)>K}] = 0.$$

(iii) $\liminf_{t\to\infty} \frac{t\Lambda(x,t)}{\log t} > 0$, where $\Lambda(x,t) = \sup_{\theta \in \mathcal{R}}[\theta x - \frac{1}{t}\log E[e^{\theta A(t)}]]$.

From the controlled flows point of view, the system we have described above can be thought of as a closed loop system with delay, and feedback control applied at the routers based on *queue based marking function* denoted by $p_q(\cdot)$. A popular modeling and analysis methodology for such closed-loop systems in the Internet context has been through functional differential equations based *fluid models*.

The generic model of such a system consists of a collection of user flows, a router modeled by marking functions which signal congestion by marking flows, and receivers which detect the marks and informs the respective flows to increase or decrease their transmission rate. We model flows by fluid processes. We denote the fluid rates of individual flows in the n-th system by $\{x_n^i(t), i = 1, \ldots, n\}$, where $x_n^i(t)$ denotes the transmission rate of a controlled flow i at time t. The dynamics of the transmission rate for each user are governed by a differential equation based controller as discussed in Section I. We comment that the controller in (2) is called a proportional-fair controller [12], as controllers of this form lead to a proportionally-fair allocation of bandwidth across users. The results in this paper, however, apply to any differential equation based congestion controller as long as $\dot{x}_n^i(\cdot)$ is bounded (i.e., the transmission rate is Lipschitz). In particular, suppose that the transmission rate $x_n^i(\cdot)$ is bounded by some constant L. This in-turn implies that $x_n(\cdot)$ is Lipschitz continuous with some parameter $M < \infty$ [14]. In the rest of this paper, we assume that the transmission rate is Lipschitz continuous with parameter M.

Let $A_n(t) = \sum_i A_n^i(t)$ be the *cumulative number of arrivals* until time t due to *uncontrolled flows*, and $X_n(t) = \sum_i x_n^i(t)$ be the total *arrival rate* at time t due to controlled flows. From Assumption 2.1, $E(A_n(t)) = n\lambda t$.

For the *controlled flows*, let us denote the average arrival rate by

$$x_n(t) = \frac{1}{n}X_n(t).$$

Further, we define the total volume of arrivals (due to the controlled flows) until time t by $Y_n(t)$, where

$$\begin{aligned} Y_n(t) &= \int_0^t X_n(z)dz \\ &= n\int_0^t x_n(z)dz \end{aligned} \tag{3}$$

Finally, we assume that the initial conditions satisfy

$$\begin{aligned} x_n^i(0) &\xrightarrow{n\to\infty} x^i(0) \\ x_n(0) &\xrightarrow{n\to\infty} x(0) \\ Q_n(0) &\xrightarrow{n\to\infty} Q(0) < \infty \\ x(0)+\lambda &< c \end{aligned} \tag{4}$$

Heuristically, these conditions correspond to the assumption that the initial condition is well defined, and is a stable system.

III. Limiting Rate Based Marking Function

For a fixed T, we are interested in studying the queue length process $Q_n(t)$ over the time-interval $[0, \frac{T}{n}]$. Thus, we are interested in the queueing behavior at the router over a short interval of time. Even over this small time interval, we will show that the the queue reaches "steady-state" behavior. This occurs due to the fact that the capacity is very large (nc), and causes the queue to "regenerate" an arbitrarily large number of times over the interval $[0, \frac{T}{n}]$. However, from a single *end-system* (the user) point of view, this corresponds to a very short interval of time. Thus, one can expect that the end-user will only perceive the statistical "steady-state" queueing behavior. The following sections quantify the above heuristic.

For any $s \in [0, \frac{T}{n}]$, we have the following queue length process

$$\begin{aligned} Q_n(s) &= \sup_{r\in[0,s]} [A_n(r) + Y_n(r) - ncr + Q_n(0)] \\ &= \sup_{r\in[0,s]} \left[A_n(r) + n\int_0^r x_n(z)dz - ncr + Q_n(0)\right] \end{aligned}$$

Now, let us study the processes (X_n, Y_n, A_n, Q_n) over a *slowed-down* time-scale. In other words, for $t \in [0,T]$, we define the processes

$$\begin{aligned} q_n(t) &= Q_n\left(\frac{t}{n}\right) \\ a_n(t) &= A_n\left(\frac{t}{n}\right) \\ y_n(t) &= Y_n\left(\frac{t}{n}\right) \end{aligned}$$

Thus, we have for any $s \in [0,T]$,

$$\begin{aligned} q_n(s) &= Q_n\left(\frac{s}{n}\right) \\ &= \sup_{\frac{r}{n}\in[0,\frac{s}{n}]} \left[A_n\left(\frac{r}{n}\right) + Y_n\left(\frac{r}{n}\right) - \frac{ncr}{n} + Q_n\left(\frac{0}{n}\right)\right] \\ &= \sup_{r\in[0,s]} [a_n(r) + y_n(r) - cr + q_n(0)] \end{aligned} \tag{5}$$

By assumption, each individual data rate ($x_n^i(t)$) is Lipschitz continuous with some parameter $M < \infty$. This also implies that the average data rate ($x_n(r)$) is Lipschitz continuous with parameter M. Let us now define

$$\tilde{q}_n(s) = \sup_{r\in[0,s]} [a_n(r) + rx(0) - cr + q(0)]$$

Lemma 3.1: Given $\epsilon > 0$, we can find N such that $\forall n > N$,

$$||q_n(t) - \tilde{q}_n(t)|| < \epsilon, \tag{6}$$

where $||\cdot||$ is the Skorohod metric [15] in the space $\mathcal{D}([0,T] : \mathcal{R}^+)$.

Proof: The proof is presented in the Appendix. ∎

Lemma 3.2: Suppose that $a_n(s) \to a(s)$ in the space $\mathcal{D}([0,T] : \mathcal{R}^+)$. Then, given any $\epsilon > 0$, there exists N such that $\forall n > N$ we have

$$||q(s) - \tilde{q}_n(s)|| < \epsilon, \tag{7}$$

where $q(s)$ is defined by

$$q(s) = \sup_{r\in[0,s]} [a(r) + rx(0) - cr + q(0)], \tag{8}$$

and $a(s)$ is a Poisson process with arrival rate λ.

Proof: The proof is presented in the Appendix. ∎

We now show that the queue length process over the slowed-down time-scale converges weakly to the queue length process of a M/D/1 queue with service rate $c - x(0)$. In [10], the authors showed a similar result for the stationary distribution of the queue. In this paper, we are interested in the *path properties of the queue* because the marks received by the end-user depends on the integral of the marking function over the (unscaled) time interval $[0, T/n]$. Thus, it is not sufficient for us to consider only the stationary distribution. We show that the slowed-down queue length process converges to the corresponding M/D/1 queueing process "uniformly" (to be precise, with respect to the Skorohod metric) over the time interval $[0,T]$.

Theorem 3.1: As $n \to \infty$, we have

$$q_n(t) \xrightarrow{w} q(t), \quad s \in [0,T] \quad \text{over} \quad \mathcal{D}([0,T] : \mathcal{R}^+)$$

where $\xrightarrow{w}$ represents weak convergence, and $q(t)$ is the queue-length process of a single server M/D/1 queue, with deterministic service rate $c - x(0)$, and arrival process $a(t)$, which is a Poisson process of rate λ.

Proof: From the superposition theorem for point processes [13], we know that a_n converges weakly to a Poisson process with rate λ denoted by $a(t)$ in $\mathcal{D}([0,\infty) : \mathcal{R}^+)$. From the Skorohod representation theorem [15], we can find processes $a_n'(t)$ and $a'(t)$ in $\mathcal{D}([0,\infty) : \mathcal{R}^+)$ such that

$$a_n(t) \stackrel{\text{dist}}{=} a_n'(t)$$

$$a(t) \overset{\text{dist}}{=} a'(t),$$

where $\overset{\text{dist}}{=}$ means "equivalence in distribution" and

$$||a'_n(t) - a'(t)|| \xrightarrow{n\to\infty} 0 \quad \text{in} \quad \mathcal{D}([0,\infty) : \mathcal{R}^+) \tag{9}$$

Corresponding to the arrival processes $a'(t)$ and $a'_n(t)$, let us define $q(t)$, $q'_n(t)$, and $\tilde{q}'_n(t)$ by Equations (8), (5), and (6) respectively.

Then, it suffices to prove that $\forall \epsilon > 0$, we can find N such that $\forall n > N, ||q'_n(t) - q'(t)|| < \epsilon$ in the space $\mathcal{D}([0,\infty) : \mathcal{R}^+)$. By triangle inequality of Skorohod norm,

$$||q'_n(t) - q'(t)|| \le ||q'_n(t) - \tilde{q}'_n(t)|| + ||\tilde{q}'_n(t) - q'(t)||$$

By applying Lemma 3.1 to the first term of RHS and Lemma 3.2 to the second term of RHS, the result follows. ■

Using this result, we now show that the total volume of marks received over the (slowed-down) time-interval $[0,T]$ converges that given by an M/D/1 queue.

Theorem 3.2: Suppose that

$$q_n(t) \quad \xrightarrow{w} \quad q(t), \quad s \in [0,T] \quad \text{over} \quad \mathcal{D}([0,T] : \mathcal{R}^+),$$

where $q_n(t)$ and $q(t)$ is defined as (5) and (8). Then, we have

$$\int_0^T p_q(q_n(y))\, dy \quad \xrightarrow{w} \quad \int_0^T p_q(q(y))\, dy \tag{10}$$

$$\int_0^T x_n^i(y/n) p_q(q_n(y))\, dy \quad \xrightarrow{w} \quad \int_0^T x^i(0) p_q(q(y))\, dy \tag{11}$$

Proof: From Theorem 3.1 and Skorohod representation theorem, we can find q'_n and q' in $\mathcal{D}([0,\infty) : \mathcal{R}^+)$ such that q'_n converges to q' in the Skorohod topology. By the definition of convergence in the Skorohod topology, we can find a strictly increasing, continuous function λ_n of $[0,T]$ onto itself and $N_1 > 0$ such that for a given $\epsilon > 0$, and $\forall n > N_1$,

$$\begin{aligned} \sup_{t\in[0,T]} |q'_n(\lambda_n(t)) - q'(t)| &< \epsilon \\ \sup_{t\in[0,T]} |\lambda_n(t) - t| &< \epsilon \end{aligned} \tag{12}$$

By adding and subtracting a common term, we have

$$\begin{aligned} &\left| \int_0^T p_q(q'_n(y))\, dy - \int_0^T p_q(q'(y))\, dy \right| \\ \le &\left| \int_0^T p_q(q'_n(y))\, dy - \int_0^T p_q(q'_n(\lambda_n(y)))\, dy \right| \\ &+ \left| \int_0^T p_q(q'_n(\lambda_n(y)))\, dy - \int_0^T p_q(q'(y))\, dy \right| \end{aligned}$$

For the second term of RHS, using Lipschitz continuity assumption of p_q and the condition (12),

$$\begin{aligned} &\left| \int_0^T p_q(q'_n(\lambda_n(y)))\, dy - \int_0^T p_q(q'(y))\, dy \right| \\ &\le \int_0^T L_q |q'_n(\lambda_n(y)) - q'(y)| < L_q T \epsilon, \end{aligned} \tag{13}$$

where $0 < L_q < \infty$ is the Lipschitz constant of $p_q(\cdot)$.

Next, for the first term of RHS, We know that $q'(s) \in \mathcal{D}([0,T] : \mathcal{R}^+)$ has a finite number of jumps denoted by $\mathcal{J}(q') < \infty$, since the arrival process is a Poisson process with a finite rate over the finite interval of time $[0,T]$. From the condition (12), we can find $N_2 > 0$ such that $\forall n > N_2$, $J \triangleq \mathcal{J}(q'_n) = \mathcal{J}(q')$. Let us denote the jump times of $q'_n(s)$ by $\{t_n^j, j = 1,\ldots,J\}$.

Now, we divide the entire interval $[0,T]$ into two sets of intervals A_1 and A_2, where $A_1 = \{I_j \triangleq [t_n^j - \epsilon, t_n^j + \epsilon], j = 1,\ldots,J\}$ and $A_2 = [0,T]\backslash A_1$. By taking $\epsilon < 0.5\min\{t_n^1, t_n^2 - t_n^1, \ldots, T - t_n^J\}$, this ensures that there is only one jump of the processes, $q'_n(\lambda_n(s))$ and $q'_n(s)$ in the interval I_j. From Lipschitz continuity of p_q, $\forall s \in [0,T]$,

$$|p_q(q'_n(\lambda_n(s))) - p_q(q'_n(s))| \quad \le \quad L_q |q'_n(\lambda_n(s)) - q'_n(s)|$$

Let $N_{max} = \max(N_1, N_2)$. Then, $\forall n > N_{max}$,

$$|q'_n(\lambda_n(s)) - q'_n(s)| \le \begin{cases} 1 & \text{if} \quad s \in A_1, \\ \epsilon(c - x(0)) & \text{if} \quad s \in A_2. \end{cases}$$

Thus,

$$\begin{aligned} &\left| \int_0^T p_q\left(q'_n(y)\right)\, dy - \int_0^T p_q\left(q'_n(\lambda_n(y))\right)\, dy \right| \\ \le &\left| \int_{A_1} p_q\left(q'_n(y)\right)\, dy - \int_{A_1} p_q\left(q'_n(\lambda_n(y))\right)\, dy \right| \\ &+ \left| \int_{A_2} p_q\left(q'_n(y)\right)\, dy - \int_{A_2} p_q\left(q'_n(\lambda_n(y))\right)\, dy \right| \\ \le &\int_{A_1} \left| p_q\left(q'_n(y)\right) - p_q(q'_n(\lambda_n(y))) \right| dy \\ &+ \int_{A_2} \left| p_q\left(q'_n(y)\right) - p_q(q'_n(\lambda_n(y))) \right| dy \\ \le &2L_q J\epsilon + L_q(c - x(0))\epsilon(T - 2J\epsilon) \\ = &\epsilon\left(2L_q J + L_q(c - x(0))(T - 2J\epsilon)\right) \end{aligned} \tag{14}$$

Since ϵ is arbitrary in (13) and (14), this completes the proof. The proof of (11) is analogous. ■

A. An Equivalent Rate Based Marking Function

Let us consider the marks received over the time-interval $[0, \frac{T}{n}]$ by some user i. By definition, *the marked volume of data* over this time-interval is given by

$$\int_0^{T/n} x_n^i(y) p_q(Q_n(y))\, dy \quad = \quad \frac{1}{n}\int_0^T x_n^i(y/n) p_q(q_n(y))\, dy$$

Thus, *the time-average volume of marks* received by user i over the time-interval $[0, \frac{T}{n}]$ is

$$\begin{aligned} &\frac{1}{T/n}\int_0^{\frac{T}{n}} x_n^i(y) p_q(Q_n(y))\, dy = \\ &\qquad \frac{1}{T}\int_0^T x_n^i(\frac{y}{n}) p_q(q_n(y))\, dy \end{aligned} \tag{15}$$

Thus, from Theorem 3.2, we have

$$\begin{aligned} &\frac{1}{T/n}\int_0^{\frac{T}{n}} x_n^i(y) p_q(Q_n(y))\, dy \to \\ &\qquad x^i(0)\frac{1}{T}\int_0^T p_q(q(y))\, dy \end{aligned} \tag{16}$$

where $q(y)$ is the queue-length process of an M/D/1 queue with Poisson arrival rate λ and capacity $c - x(0)$. Let us define

$$p_T(x(0)) \triangleq \frac{1}{T} \int_0^T p_q(q(y))\, dy. \tag{17}$$

For n large enough, we see from (16) that the interaction between the router queuing process and the congestion controller at a fixed user occurs *only through* this function $p_T(\cdot)$.

Further, we observe that $q(y)$ is a regenerative process when $\frac{\lambda}{c-x} < 1$ and $x < c$. Thus, from the ergodic theorem for a regenerative process [16] and Smith's theorem [17],

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T p_q(q(y))\, dy \quad = \quad E_{\pi^x}[p_q(Q)], \tag{18}$$

where π^x is the stationary distribution of an M/D/1 queue with arrival rate λ and capacity $(c - x)$.

For T large enough, and by defining

$$p(x) \quad = \quad \begin{cases} E_{\pi^x}[p_q(Q)] & \text{if} \quad \frac{\lambda}{c-x} < 1 \quad \text{and} \quad x < c, \\ 1 & \text{if} \quad x \geq c \quad \text{or} \quad \frac{\lambda}{c-x} \geq 1. \end{cases}$$

we see from (17) and (18) that the congestion controller dynamics with a queue-based marking function $p_q(\cdot)$ can be well approximated by a *equivalent* system with only a rate-based controller $p(x)$ at the router, where x is simply the average arrival rate from the controlled flows (averaging over flows, not time) to the router queue.

IV. Numerical and Simulation Results

A. Limiting Rate Based Marking Function: REM

In this section, we validate our analysis of the limiting rate based marking function in case of REM controller [9].

First, from the P-K formula for stationary workload V of an M/D/1 queue [18], we have

$$E[e^{-sV}] \quad = \quad \frac{1-\rho}{1 - \frac{\lambda}{s}(1 - e^{-s/\mu})}, \tag{19}$$

where μ is the service rate, λ is the arrival rate, and $\rho = \lambda/\mu$. From [9], the queue based marking function of REM controller is defined as

$$p_q^{rem}(Q) \quad = \quad 1 - e^{-\alpha Q}, \tag{20}$$

where α is a suitable constant pre-defined in the system, and Q is the *queue length* in the system. Further, for the fluid queueing system we consider, it follows from the definition of workload [18] that $V = Q/c$. Thus, we have

$$\begin{aligned} p(x) \quad &= \quad E_{\pi^x}[p_q^{rem}(Q)] \\ &= \quad E_{\pi^x}[1 - e^{-\alpha c V}] \\ &= \quad 1 - \frac{1-\rho}{1 - \frac{\lambda}{\alpha c}\left(1 - e^{-\alpha c/(c-x)}\right)}, \end{aligned} \tag{21}$$

where $\rho = \frac{\lambda}{c-x}$. In the next section, we compare simulation results with queue based marking with REM and compare that to numerical results using its equivalent rate based marking function given by (21).

B. Simulation Results

First, we describe the simulation environments used in this section. The network topology adopted in the simulation is the same as that introduced in Section I. There is only one queue with FIFO queueing discipline accessed by n controlled and uncontrolled flows, and the system capacity is $100 \times n$ packets per second. We use a discrete-time version of the proportionally fair controller [12] described by the following difference equation:

$$x_n^i[k+1] \quad = \quad u\kappa\left(w - x_n^i[k-d]p_q\left(Q_n[k-d]\right)\right),$$

where u is the update interval, and d is the round-tip propagation delay. In our simulations, $w = 5$, and we use a value of 100 msec as the round-trip delay, and 100 msec as the update interval.

Uncontrolled flows are modeled by ON-OFF processes [19], where ON and OFF period is exponentially distributed with rate 0.2 and the packet transmission rate in the ON period is Poisson with a suitable mean so that the load due to the uncontrolled flows is a fixed fraction of the link capacity.

Table I and Table II shows the simulation results for two cases. In Table I, we consider the case where the uncontrolled flows occupy 35% of the link bandwidth. The table indicates the number of flows, time average of the total rate as well as the time average due to controlled flows from the simulation. The numerical result based on the equivalent rate based marking function (21) is also indicated in the table. The entry ER corresponds to the equilibrium rate from the proportionally fair controller with marking at the router according to (21). The results show that there is about a 6% difference between the numerical and simulation results. This is small, especially considering the fact that we are comparing the equilibrium rate from the numerical analysis with the time-average of the simulation, which has fluctuations due to the ON-OFF uncontrolled flows, as well as the randomness introduced due to probabilistic marking [6] in the simulations. In general, we have observed that the fluid equilibrium rate falls within the "stochastic fluctuations" in the simulations.

Similar results are presented in Table II, we consider the case where the uncontrolled flows occupy 50% of the link bandwidth. The difference between the numerical and simulation results in this case is only 3%. Thus, these results suggest from a user perspective, the equivalent marking function approximation that we derived in this paper predicts the queueing dynamics at the router accurately. This approximation suggests a fast simulation method for networks where we replace queues at routers by their equivalent marking function. Future work will focus on developing such a simulator, and studying its performance.

Appendix

Proof of Lemma 3.1

Proof: Let us denote $b_n(r) = a_n(r) - cr$, $c_n(r) = y_n(r) + q_n(0)$, and $c = rx(0) + q(0)$. Then,

$$\sup_{t\in[0,T]} |q_n(t) - \tilde{q}_n(t)|$$

TABLE I
UNCONTROLLED TRAFFIC OCCUPIES 35% OF THE LINK. ALL NUMBERS IN PACKETS PER SECOND. N: NUMBER OF FLOWS, AT: TIME-AVERAGE OF TOTAL RATE (CONTROLLED + UNCONTROLLED), AC: TIME-AVERAGE OF CONTROLLED FLOW RATE, ER: EQUILIBRIUM RATE FOR CONTROLLED FLOWS USING EQUIVALENT RATE BASED MARKING FUNCTION.

n	AT	AC	ER
10	684.511919	333.796540	380
25	1676.073333	801.183131	950
50	3334.636667	1573.633889	1900
75	4981.349697	2356.113434	2850
100	6636.427677	3129.419192	3800

TABLE II
UNCONTROLLED TRAFFIC OCCUPIES 50% OF THE LINK. ALL NUMBERS IN PACKETS PER SECOND. N, AT, AC, AND ER ARE THE SAME AS IN TABLE I

.

n	AT	AC	ER
10	746.176364	253.837753	277.5
25	1858.050404	614.868283	687.5
50	3709.321111	1211.174545	1375
75	5559.191616	1808.234495	2062.5
100	7406.805152	2407.880795	2775

$$
\begin{aligned}
&= \sup_{t\in[0,T]} \left| \sup_{r\in[0,t]} \big[b_n(r)+c_n(r)\big] - \sup_{r\in[0,t]} \big[b_n(r)+c\big] \right| \\
&= \sup_{t\in[0,T]} \left| \sup_{r\in[0,t]} \big[b_n(r)+c-c+c_n(r)\big] - \right. \\
&\qquad \left. \sup_{r\in[0,t]} \big[b_n(r)+c\big] \right| \\
&\le \sup_{t\in[0,T]} \left| \sup_{r\in[0,t]} \big[b_n(r)+c)\big] + \sup_{r\in[0,t]} \big[c_n(r)-c\big] - \right. \\
&\qquad \left. \sup_{r\in[0,t]} \big[b_n(r)+c\big] \right| \\
&= \sup_{t\in[0,T]} \left| \sup_{r\in[0,t]} \big[c_n(r)-c\big] \right| = \left| \sup_{r\in[0,T]} \big[c_n(r)-c\big] \right| \\
&= \left| \sup_{r\in[0,T]} \big[y_n(r)-rx_n(0)+rx_n(0)-rx(0) \right. \\
&\qquad \left. +q_n(0)-q(0)\big] \right| \\
&\le \left| \sup_{r\in[0,T]} \big[y_n(r)-rx_n(0)\big] \right| + \left| \sup_{r\in[0,T]} \big[rx_n(0)-rx(0)\big] \right. \\
&\qquad + \left| \sup_{r\in[0,T]} \big[q_n(0)-q(0)\big] \right| \\
&\le \left| \sup_{r\in[0,T]} \big[y_n(r)-rx_n(0)\big] \right| + T|x_n(0)-x(0)| \\
&\qquad +|q_n(0)-q(0)| \qquad (22)
\end{aligned}
$$

Since $|x_n(0)-x(0)| \to 0$ and $|q_n(0)-q(0)| \to 0$ from the assumption (4), it suffices to prove that as $n \to \infty$,

$$
\left| \sup_{r\in[0,T]} \big[y_n(r)-rx_n(0)\big] \right| \quad \to \quad 0
$$

For any $r \in [0,T]$, we have

$$
\begin{aligned}
\left| n\int_0^{r/n} x_n(z)dz - n\int_0^{r/n} x_n(0)dz \right| &\le n\int_0^{r/n} Mzdz \\
&\le \frac{Mr^2}{2n}
\end{aligned}
$$

Thus, by definition of $y_n(r)$, we have

$$
\sup_{r\in[0,T]} |\, y_n(r) - rx_n(0) \,| \quad \le \quad \frac{MT^2}{2n} \qquad (23)
$$

From (23) and (22), we can find N such that $\forall n > N$, $\sup_{t\in[0,T]} |q_n(t) - \tilde{q}_n(t)| < \epsilon$. Also, the fact that convergence with respect to uniform topology implies convergence with respect to Skorohod topology completes the proof. ■

Proof of Lemma 3.2

Proof: By definition of convergence in Skorohod topology, for a given $0 < \epsilon < 1$ and sufficient large n, we can find a strictly increasing, continuous function λ_n of $[0,T]$ onto itself such that

$$
\begin{aligned}
\sup_{s\in[0,T]} |a_n(\lambda_n(s)) - a(s)| &< \epsilon \\
\sup_{s\in[0,T]} |\lambda_n(s) - s| &< \epsilon, \qquad (24)
\end{aligned}
$$

and for n large enough, we have

$$
J \triangleq \mathcal{J}(a) = \mathcal{J}(a_n), \qquad (25)
$$

where $\mathcal{J}(x)$ is the number of jumps over the space $\mathcal{D}([0,T] : \mathcal{R}^+)$. The fact that $a(s), s \in [0,T]$ is a Poisson process with a finite rate ensures that we have a finite number of jumps almost surely over the finite interval of time. In addition, as any "extra jump" would lead to a distance of 1, which contradicts to the condition (24), (25) follows. Let us denote the arrival times of a and a_n by $\{t^j, j = 1, \ldots, J\}$ and $\{t_n^j, j = 1, \ldots, J\}$, respectively. We know that the arrival times of $a(t)$ ($a_n(t)$) are equivalent to the jump times of $q(t)$ ($q_n(t)$). Also, we know that

$$
\sup_{1\le i\le J} |t_n^i - t^i| < \epsilon,
$$

for sufficient large n from (24). For a function $\lambda_n(s)$ satisfying (24), we choose a piece-wise linear function such that $\lambda_n(t^i) = t_n^i, i = 1, \ldots, J$. Obviously, $\lambda_n(s)$ is a continuous and strictly increasing function over the interval $[0,T]$. To complete the proof, it suffices to show that $\sup_{s\in[0,T]} |q_n(\lambda_n(s)) - q(s)|$ is arbitrarily small. Let

$$
\Delta q_n(s) = |q_n(\lambda_n(s)) - q(s)|
$$

Then, we have the following recurrence relation

$$
\Delta q_n(t^{j+1}) \le \Delta q_n(t^j) + \epsilon(c - x(0)),
$$

since $q_n(\lambda_n(s))$ and $q(s)$ has only one jump at time t^{j+1} in the interval $(t^j, t^{j+1}]$. Thus, the queue size difference is only that due to the amount of service with rate $c - x(0)$ over the time difference $|\lambda_n(s) - s|$. Further, for any $r \in [t^j, t^{j+1}]$, we have

$$
\Delta q_n(r) = \max[q_n(t^j), q_n(t^{j+1})],
$$

as the $q_n(\cdot)$ and $q(\cdot)$ are piece-wise linear between jumps. Thus, it is enough to check only the jump times of the process $q(s), s \in [0, T]$. Thus,

$$\begin{aligned} \sup_{s \in [0,T]} |q_n(\lambda_n(s)) - q(s)| &\leq \max_{1 \leq j \leq J} \Delta q_n(t^j) \\ &\leq \epsilon J(c - x(0)), \end{aligned}$$

this completes the proof. ■

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