On the Progressive Spread over Strategic Diffusion: Asymptotic and Computation*

Jungseul Ok Jinwoo Shin Yung Yi†

***Abstract*—We study how an innovation (e.g., product or technology) diffuses over a social network when individuals strategically make selfish, rational choices in adopting the new innovation. This diffusion has been studied by modeling individuals' interactions with a noisy best response dynamic over a networked coordination game, but mainly in the non-progressive setup. In this paper, we study the case when people are progressive, i.e., never going back to the old technology once the new technology is chosen, where such a progressive behavior is explained using the notion of *sunk cost fallacy* in social psychology. Our main focus is on the diffusion time, i.e., time till all choose the new innovation. To this end, we first provide a combinatorial characterization of the diffusion time that corresponds to the time reaching the absorbing state in a Markov chain. Based on this, we propose a polynomial-time algorithm that computes the diffusion time, where such a task is known to be computationally intractable in the non-progressive diffusion. Second, we asymptotically quantify the diffusion times for a class of well-known social graph topologies, and compare them to those under the non-progressive diffusion. Finally, we study the impact of seeding to speed up the diffusion in the progressive setup, and show that the diffusion speed is impossible to significantly accelerate with just a small-budget seeding, which is in part in stark contrast to that in the non-progressive diffusion. Our results provide not only understandings on the progressive strategic diffusion in a social network, but also computational tractability on other related problems, e.g., seeding, which we believe should be of broader interest in the future.**

I. Introduction

Social networks are major routes, explicitly and implicitly, for most individuals exchanging their opinions about new products, social trends and political issues via their interactions. The motivation of this paper is to study the diffusion of technology innovation over social networks, where people adopt the new technology based on the strategic decision of whether the new technology gives them higher payoff or not. In this case, the payoff of each individual is quantified by the summation of payoffs between her and each of her neighbors (or friends). For example, consider an old OS, and a new OS is released to the market (e.g., Windows 7 and Windows 8). Due to the issues of software compatibility and innovation of the new OS, one gets higher payoffs when she can "coordinate" with more social neighbors, i.e., use the same OS, and the coordination with the new OS provides higher payoff than that with the old OS. This game-based diffusion model has been studied with various research goals, see e.g., [1]–[9], where of primary interest is the diffusion time – the time until all individuals finally choose the new technology.

Most prior work on this line has been mainly based on the assumption of *non-progressiveness* that one is allowed to freely switch between old and new technologies multiple times. However, *progressive* behaviors are often observed in practice, i.e., once people adopt a new technology, they are frozen and do not return to an old technology. For example, most people do not switch back to the old OS after installing the new OS even if they suffer from software incompatibility issues with friends who use old OS yet. In social psychology [10], such progressive behavior is explained as the so-called *sunk cost fallacy* where individuals have strong misgivings to "waste" investment.

Motivated by this, the main goal of this paper is to understand the diffusion time in the game-based diffusion model under the progressive assumption, i.e., how it behaves differently compared to that under the non-progressive one studied in the literature. Our contributions are summarized in what follows.

Characterization and computation of diffusion time. We first characterize the diffusion time in the progressive setup as a convenient combinatorial value, which is analogous to the known combinatorial formula in the non-progressive one [4]. Somewhat surprisingly, we found that our combinatorial characterization is computationally tractable, while that in [4] is not. Namely, we develop a polynomial-time algorithm that computes the diffusion time in the progressive setup exactly, being in sharp contrast to the non-progressive setup whose diffusion time is known to be computationally intractable (i.e., NP-hard). Such a polynomial-time computation of the diffusion time is a quite unique characteristic in the progressive diffusion, and can be also useful for other related problems, e.g., we also design a polynomial-time seeding algorithm as we provide its details later.

Asymptotic analysis of diffusion time. Second, we establish the asymptotic bounds on the diffusion time in the progressive setup for the following social network topologies: d-dimensional, random k-regular, small-world, scale-free, and Erdős-Rényi graphs. From this analysis, we asymptotically quantify the positive impact of progressiveness on the diffusion speed in the game-based model, briefly summarized as follows. The diffusion time is significantly reduced compared to that in the non-progressive case, but well-connected graphs such as Erdős-Rényi graphs still experience slow diffusion. An interesting consequence of our results is that in random k-regular

∗ : This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2013R1A2A2A01067633).

†: The authors are with the Department of Electrical Engineering, KAIST, South Korea, {ockjs,jinwoos,yiyung}@kaist.ac.kr

and small-world graphs, the diffusion time is independent of the network size, while it is not the case in the non-progressive setup. In addition, we found a new insight for scale-free graphs that a more skewed degree distribution leads to higher impact of progressiveness on the diffusion time.

Acceleration of diffusion time. Third, we study the effect of seeding for accelerating the diffusion speed for the graphs having long diffusion time, where seeding means that a subset of the nodes pre-adopt the new technology a priori.[1] We prove that the seeding effect in the progressive setup is marginal for a small seed budget in the sense that seeding of a constant number of nodes can reduce the diffusion time by only a constant factor, implying that an order-wise reduction in diffusion time requires a budget scaling with the network size. This is again quite different from in the non-progressive setup, where there exist some cases when the diffusion time is drastically reduced even with a single seeding (see Section III-C). Nevertheless, given a social network and a seed budget, we also propose a simple greedy algorithm choosing seeds which outperforms other popular algorithms in our experiments (see Section V). Such a low-complexity design is possible mainly due to our polynomial computation of the progressive diffusion time as we explained earlier.

A. Related Work

In what follows, we briefly summarize the related work to our results. First, we note that research on diffusion in the progressive, game-based model is quite limited to the best of our knowledge, which includes [1], [14], [15]. The authors of those papers study the conditions (of network topology and the payoff difference between old and new technologies) on the existence of a finite size of seed set, referred as the so-called "contagion set," that enables the entire spread in the infinite network. In particular, the authors of [1] study the case of the progressive mode. Our work differs from them in that we consider a noisy best response dynamic, but more importantly, our major focus is on the diffusion time that is a significant quantity in the diffusion study. Despite of the difference of interests, our result can be used to determine the existence of contagion set (see Section III-C). Related to seeding, in the non-progressive setup, influence maximization via seeding has been studied by [5], [16]. In particular, the authors of [16] study maximizing the influence spread by choosing a seed set against a set of "negative individuals" who refuse new technology no matter what their neighbors use.

Different from the game-based strategic diffusion as in this paper, people have extensively studied the diffusion effects with close relevance to raging epidemic, e.g., SIRS model [17], and interacting particle system, e.g., Ising model [18]. The key difference of epidemic-based model from the game-based one lies in the fact that nodes are "infected" just based on the individuals' simple contacts. Popular epidemic models on which an extensive array of diffusion research has been conducted (see e.g., [17], [19]–[24], just name a few) include SI (Susceptible-Infected), SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed), IC (Independent Cascade), LT (Linear Threshold) [22], [25], [26]. The epidemic models of SI, IC and LT implicitly and explicitly include the context of progressiveness, because, for example, in the SI model, a node cannot become susceptible once it is infected. In IC and LT models, a node has just one chance of infecting other nodes and cannot infect others if such chance passed. SIS model is clearly non-progressive, since nodes can freely switch back to the susceptible state. SIR model has arguably partial progressiveness in the sense that individuals are allowed to turn back to the old technology (i.e., susceptible) after choosing the new technology (i.e., infected), but are frozen thereafter (i.e., removed with immunity). In [22], [27], the authors addressed the influence maximization problem in LT and IC models.

In summary, the social diffusion problems have been heavily studied in the literature under various assumptions and setups, while this paper first tackles the asymptotic and computational aspects of the progressive diffusion time in game-based models. We believe that our results should be of boarder interest for other related problems, e.g., seeding and contagion set, in the future research.

Organization. The rest of this paper is organized as follows: In Section II, we describe our diffusion model of interest with necessary preliminaries. Then, in Section III, we present our main results on the diffusion time analysis in the progressive setup, followed by the proofs in Section IV. In Section V, we provide the experimental results, and conclude in Section VI.

II. Model and Preliminaries

A. Networked Coordination Game

We model a social network with an undirected graph $G = (V, E)$ with $|V| = n$, where V and E are the sets of nodes and edges, respectively. Each node $i \in V$ represents an individual (or a user) and has its strategy $x_i \in \{-1, +1\}$, where $+1$ or -1 represent the new and old technologies, respectively. Each edge represents a social relationship between two individuals. Let $N(i)$ denote the set of node i's neighbors, i.e., $N(i) = \{j \in V \mid (i, j) \in E \text{ and } i \neq j\}$.

We now describe the payoffs of individuals, where an individual's payoff is affected by its neighbors' strategies. To model it, we first consider the well-known two-person coordination game whose payoff matrix is given by Table I, where an individual can choose one of new or old technologies, $+1$ and -1. We make the following mild assumptions on the payoffs. First, there always exists coordination gain, i.e., $a > d$ and $b > c$. Second, coordination gain becomes larger for the new technology, i.e., $a - d > b - c$.

TABLE I
PAYOFF MATRIX OF TWO-PERSON COORDINATION GAME

	$+1$	-1
$+1$	(a, a)	(c, d)
-1	(d, c)	(b, b)

[1]For example, a software company with a new OS release may give economic incentives to a group of people whose task is to use the new OS in advance and advertise it to their social neighbors. Seeding is motivated at other applications, e.g., [11] in viral marketing, [12] in graph detection, and [13] in computer virus vaccine dissemination.

The two-person coordination game is extended to a networked version as follows: We let $\boldsymbol{x} = (x_j \in \{-1, +1\} : j \in V)$, and $\boldsymbol{x}_{-i} = (x_j : j \in V \setminus \{i\})$ be a strategy vector chosen by the entire nodes and those except for i, respectively. Then, in the n-person game over G, node i's payoff $P_i(x_i, \boldsymbol{x}_{-i})$ for the state $\boldsymbol{x}$ is modeled to be the aggregate payoff against all of i's neighbors, i.e.,

$$P_i(x_i, \boldsymbol{x}_{-i}) \quad = \quad \sum_{j \in N(i)} P(x_i, x_j), \tag{1}$$

where $P(x_i, x_j)$ is the payoff from the two-person coordination game in Table I. For notational convenience, let $\boldsymbol{-1}$ (resp. $\boldsymbol{+1}$) denote the state where every user adopts -1 (resp. $+1$).

B. Diffusion Dynamics

In this section, we describe the diffusion dynamics of our interest: how new technologies are spread over the social network G under strategic choices of individuals. We assume that each individual has its own independent Poisson clock with unit rate, and whenever the clock ticks, it decides which strategy to adopt. We first describe the well-known best response dynamics, where each individual selects a strategy that maximizes its own payoff: a node i chooses $+1$, if

$$(a-d)|N^+(i)| \quad \geq \quad (b-c)|N^-(i)| \tag{2}$$

where $N^+(i)$ and $N^-(i)$ denote the sets of node i's neighbors adopting $+1$ and -1, respectively. Noting that for a given state $\boldsymbol{x}$, $P_i(+1, \boldsymbol{x}_{-i}) - P_i(-1, \boldsymbol{x}_{-i})$ is the payoff difference between when node i chooses $+1$ and -1, the best response of node i is $\text{sign}(P_i(+1, \boldsymbol{x}_{-i}) - P_i(-1, \boldsymbol{x}_{-i}))$. A simple algebra gives us the following expression of the best response of node i:

$$\text{sign}\left(h_i + \sum_{j \in N(i)} x_j \right), \tag{3}$$

where $h_i = h|N(i)|$ and $h = \frac{a-d-b+c}{a-d+b-c}$.

However, people are often affected by many external and internal noise factors in choosing their strategies. We model such "noisy" behavior by introducing small mutation probability that a strategy is irrationally chosen, often called noisy best response. A version of the noisy best response we study in this paper is *logit dynamics* [5], [6], [28]–[30]. Each person adopts a strategy according to a distribution of the logit form which allocates larger probability to those strategies delivering larger payoffs. Formally, for the given state $\boldsymbol{x}$, each node i selects the strategy $s_i \in \{-1, +1\}$ with the following probability:

$$\mathbb{P}[s_i | \boldsymbol{x}] = \frac{\exp(\beta s_i I_i(\boldsymbol{x}))}{\exp(\beta I_i(\boldsymbol{x})) + \exp(-\beta I_i(\boldsymbol{x}))}, \tag{4}$$

where

$$I_i(\boldsymbol{x}) := h_i + \sum_{j \in N(i)} x_j.$$

The parameter β represents the degree of user rationality, where $\beta = \infty$ corresponds to the best response and $\beta = 0$ lets users update their strategies uniformly at random. Note that $(a-d+b-c)s_i I_i(\boldsymbol{x})$ is the actual payoff gain for strategy s_i instead of $-s_i$ from (3), but $(a-d+b-c)$ is removed just for convenient handling of other quantities later without loss of generality.

Progressive or non-progressive. There exist two setups of individuals' updating their strategies: *non-progressive* and *progressive*. In the non-progressive setup, each node updates its strategy -1 or $+1$ upon every tick of its Poisson clock according to the transition probability in (4) with no exception, whereas in the progressive setup only nodes having -1 (i.e., the old technology) updates its strategy, i.e., once a node selects $+1$ strategy (i.e., the new technology), it is frozen. However, we note that such frozen nodes still affect other nodes having -1, and thus have impact on the entire diffusion process until all nodes adopt $+1$. In both cases, $\{\boldsymbol{x}(t) : t \geq 0\}$ forms a continuous-time Markov chain with the state space $\{+1, -1\}^n$, where $\boldsymbol{x}(t)$ denotes the (strategy) state at time t. The progressive setup is the main focus of this paper.

C. Diffusion Speed

We are interested in understanding how fast or slow every individual adopts the new technology under the strategic choices described in the previous section. To this end, we define a random variable called the *hitting time* to $\boldsymbol{+1}$ from initial state $\boldsymbol{y} \in \{+1, -1\}^n$, and denote it by $T(\boldsymbol{y})$:

$$T(\boldsymbol{y}) := \inf\{t \geq 0 \mid \boldsymbol{x}(t) = \boldsymbol{+1}, \boldsymbol{x}(0) = \boldsymbol{y}\}.$$

Using this, we next define the *typical hitting time* τ to be:

$$\tau := \sup_{\boldsymbol{y} \in \{+1,-1\}^n} \inf\left\{t \geq 0 \mid \mathbb{P}[T(\boldsymbol{y}) \geq t] \leq e^{-1}\right\}. \tag{5}$$

This means that with probability $1 - 1/e$ ($> 1/2$), every node adopts the innovation $+1$ within time τ. This typical hitting time has also been used to measure the diffusion speed for a similar model based on the close relation between hitting and mixing of the Markov chain, e.g., see [4]. Since the typical hitting time depends on the presence (or absence) of progressiveness, we let $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{N}}$ denote the typical hitting time under *progressive* and *non-progressive* setups, respectively.

The combinatorial characterization of typical hitting time $\tau_{\mathcal{N}}$ in the non-progressive setup was studied by Montanari and Saberi [4], where the authors show the following theorem.

Theorem II.1. *As $\beta \to \infty$, the typical hitting time $\tau_{\mathcal{N}}$ in the non-progressive setup is*

$$\tau_{\mathcal{N}} = \exp(2\beta\Gamma^*(G) + o(\beta)).$$

In the above, $\Gamma^*(G)$ is defined as follows:

$$\Gamma^*(G) := \max_{S_0 \subset V} \min_{\overrightarrow{v} \in \mathcal{L}(V \setminus S_0)} \max_{1 \leq t \leq |\overrightarrow{v}|} \left[H(V_t \cup S_0) - H(S_0)\right],$$

where for a subset $S \subset V$ we define $\mathcal{L}(S)$ as the set of all vertex orderings of S, and for an ordering $\overrightarrow{v} = (v_1, ..., v_{|S|}) \in \mathcal{L}(S)$, V_t is the set of nodes up to t, i.e., $V_t = \{v_1, ..., v_t\}$. Also, $H(S)$ is defined as:

$$H(S) := \text{cut}(S, V \setminus S) - h \sum_{i \in S} |N(i)|.^2 \tag{6}$$

[2]Here, $\text{cut}(A, B)$ is the number of edges between A and B, $A, B \subset V$.

The main goal of this paper is to obtain such a combinatorial characterization of typical hitting time $\tau_{\mathcal{P}}$ in the progressive setup, if possible, and understand how it behaves for various social network topologies.

III. Main Result: Progressive Diffusion

We are now ready to state the main results of this paper. First, we obtain the following theorem characterizing the typical hitting time $\tau_{\mathcal{P}}$ in the progressive setup by a min-max combinatorial optimization, which is analogous to Theorem II.1 in the non-progressive setup.

Theorem III.1. *As $\beta \to \infty$, the typical hitting time $\tau_{\mathcal{P}}$ in the progressive setup is*

$$\tau_{\mathcal{P}} = \exp(2\beta\Delta^*(G) + o(\beta)),$$

where

$$\Delta^*(G) := \min_{\overrightarrow{v} \in \mathcal{L}(V)} \max_{1 \le t \le |\overrightarrow{v}|} \left[H(V_t) - H(V_{t-1}) \right]. \quad (7)$$

The proof of Theorem III.1 is presented in Section IV-A. We refer the readers to Theorem II.1 for the definitions of $\overrightarrow{v}, V_t, \mathcal{L}(V), H(V_t)$ in the above theorem. To provide some intuition behind Theorem III.1 and II.1, consider a linear ordering $\overrightarrow{v}$ that corresponds to a diffusion path to $+\mathbf{1}$. Then, observe that $[H(V_t) - H(V_{t-1})]$ is proportional to the payoff loss of node v_t due to selecting $+1$ instead of -1. Therefore, Theorem III.1 implies that diffusion in the progressive setup typically occurs along paths that minimize the largest payoff loss of a node adopting $+1$ and the exponent of the diffusion time is dominated by the largest loss, i.e., $\Delta^*(G)$. Meanwhile in non-progressive setup, a node, who just chooses $+1$ despite of payoff loss, can easily turn back to -1 if its neighbors are still hesitating to adopt $+1$ thus diffusion in the non-progressive setup is characterized by aggregated payoff loss over a segment of the diffusion path instead of a node's loss.

A. Asymptotic Bounds on Progressive Diffusion

Theorem III.1 allows us to establish asymptotic bounds on diffusion speed for various social network topologies. In particular, we study the following representative social graphs:

- ***d-dimensional graph.*** A graph is called a d-dimensional graph with parameter R, if each node i can be embedded to a position π_i in $\mathbb{R}^d$ such that $(i,j) \in E$ implies that the Euclidean distance between π_i and π_j is less than radius R and any cube of volume B contains at most $2B$ nodes.
- ***Random k-regular graph.*** An k-regular graph is a graph where every node has the same degree k. In addition, a random k-regular graph is one selected from the uniform probability distribution over the space of all possible k-regular graphs given n nodes.
- ***Small-world graph.*** This graph has parameter (k,r), where the nodes are placed on a d-dimensional grid of side-length $n^{1/d}$. Two nodes i, j are connected by an edge if they are the nearest neighbors. Further, each node i is connected to k other nodes $j_1, j_2, ..., j_k$ drawn independently with distribution $P_i(j) \propto |i-j|^{-r}$.
- ***Scale-free graph.*** A scale-free graph with parameter γ is a graph whose degree distribution follows a power law, i.e., the fraction $f(k)$ of nodes having k connections to other nodes goes for large values of k as $f(k) = \Theta(k^{-\gamma})$.
- ***Erdős-Rényi (ER) graph.*** This graph with the parameter (n,p) is a random graph of n nodes such that every node pair becomes an edge with probability p.

For the above social graphs, we obtain the following theorem using Theorem III.1, where it characterizes asymptotic bounds on the typical hitting time in the progressive setup.

Theorem III.2. *As $\beta \to \infty$, the typical hitting time $\tau_{\mathcal{P}}$ in the progressive setup is $\tau_{\mathcal{P}} = \exp(2\beta\Delta^*(G) + o(\beta))$, where*

(a) If G is a d-dimensional graph with radius $R = O(1)$, $\Delta^*(G) = O(1)$.

(b) If G is a k-regular graph with degree $k = O(1)$, $\Delta^*(G) = O(1)$.

(c) If G is a small-world graph with shortcut edges of $k = O(1)$, $\Delta^*(G) = O(1)$.

(d) If G is a scale-free graph with parameter $\gamma > 1$, $\Delta^*(G) = O(n^{1/\gamma})$.

(e) If G is an ER graph with $np = \omega(\log n)$, $\Delta^*(G) = \Theta(np)$ with high probability.

The proof of the above theorem is presented in Section IV-B. The essence to prove the above theorem lies in obtaining the asymptotic bound on $\Delta^*(G)$ in (7) for each graph. We remark that the corresponding asymptotic bounds on $\Gamma^*(G)$ for the non-progressive setup were already established in [4] and [5], where we remind that $\Delta^*(G)$ and $\Gamma^*(G)$ are the dominant exponents of diffusion times in progressive and non-progressive setups, respectively. In Table II, we compare the diffusion speeds between both setups.[3]

TABLE II
Comparisons of diffusion speeds between progressive and non-progressive setups.

Graph	Progressive: $\Delta^*(G)$	Non-progressive: $\Gamma^*(G)$
d-dimensional $\big(R = O(1)\big)$	$O(1)$	$O(1)$ [4]
Random k-regular $\big(3 \le k = O(1)\big)$	$O(1)$	$\Omega(n)$ [4]
Small-world $\big(r < d\big)$	$O(1)$	$\Omega(n)$ [4]
Scale-free* $\big(1 < \gamma, d_{\min} \ge 2\big)$	$O(n^{1/\gamma})$	$\Omega(n)$ [4]
Erdős-Rényi $\big(np = \omega(\log n)\big)$	$\Theta(np)$	$\Theta(n^2 p)$ [5]

* Here, $d_{\min}$ is the minimum degree of the graph.

Table II implies that progressiveness in individuals' strategic choices of the new technology significantly reduces the diffusion time, where its quantitative impacts depend on the underlying social network topologies. Somewhat interestingly, in random k-regular and small-world graphs, the diffusion time in the progressive setup does not depends on the network size, while it does not hold in the non-progressive one. In addition, for scale-free graphs, one can conclude that more

[3] Additional conditions on the graph parameters unstated in Theorem III.2, but stated in Table II are needed only in the non-progressive setup.

skewed degree distribution, i.e., higher parameter γ, leads to higher impact of progressiveness.

B. Computational Complexity on Progressive Diffusion

In the previous section, we study asymptotic bounds on diffusion speed for various social networks. Another interesting question is whether one can compute the diffusion speed efficiently given a social graph, where it is known for the non-progressive setup that computing the diffusion speed, i.e., $\Gamma^*(G)$, is computationally intractable (see [4]). However, somewhat surprisingly, we show that it is not the case for the progressive setup by proposing the following polynomial-time algorithm computing the diffusion speed, i.e., $\Delta^*(G)$.

Algorithm 1: Computation of $\Delta^*(G)$

Input: Graph $G = (V, E)$
Output: Δ

Initially, $S = \emptyset$ and $\Delta = 0$.
for $t = 1, 2, ..., n$ **do**
  Pick a node $i^* \in \arg\min_{i \in V \setminus S} [H(S \cup \{i\}) - H(S)]$.
  Update Δ and S:
    $\Delta \leftarrow \max\{\Delta, H(S \cup \{i^*\}) - H(S)\}$ and then;
    $S \leftarrow S \cup \{i^*\}$.
end
Output Δ.

The above algorithm consists of n iterations which updates S and Δ in a greedy fashion. The algorithm starts with $\Delta = 0$ and $S = \emptyset$. In each iteration, a node, which minimizes a certain quantity among all $i \in V \setminus S$, is selected and added to S. The sequence of selected nodes in this way becomes an optimal ordering for $\Delta^*(G)$ as we formally state in the following theorem.

Theorem III.3. *For any given graph G, Algorithm 1 outputs $\Delta^*(G)$ in $O(n^2)$ time.*

The proof of the above theorem is presented in Section IV-C.

C. Accelerating Progressive Diffusion

In this section, we study how much one can reduce the diffusion time by choosing a subset $C \subset V$ of nodes that *initially* select new technology $+1$ (i.e., early adopters) and the diffusion dynamic starts from it. Such nodes can be interpreted as a way to encourage diffusion, where it is popularly called *seeding* and $|C|$ is often referred to as *seed budget*. Formally, seeding a subset $C \subset V$ means forcing $x_i(t) = +1$ for every node $i \in C$ and all $t \geq 0$. Now we define the typical hitting time $\tau_{\mathcal{P}}(C)$ in the progressive setup under a seeding set C and Theorem III.1 can be naturally generalized to characterize it as follows:

$$\begin{aligned}\tau_{\mathcal{P}}(C) &:= \sup_{\boldsymbol{y} \in \mathcal{S}(C)} \inf\{t \geq 0 \mid \mathbb{P}[T(\boldsymbol{y}) \geq t] \leq e^{-1}\} \\ &= \exp(2\beta\Delta^*(G; C) + o(\beta)),\end{aligned}$$

where

$$\mathcal{S}(C) := \{\boldsymbol{y} \in \{+1, -1\}^n \mid y_i = +1 \text{ if } i \in C\}$$

$$\Delta^*(G; C) := \min_{\overrightarrow{v} \in \mathcal{L}(V \setminus C)} \max_{1 \leq t \leq |V \setminus C|} [H(V_t \cup C) - H(V_{t\text{-}1} \cup C)].$$

We note that a similar extension of Theorem II.1 is doable for the non-progressive setup, where one can naturally define $\tau_{\mathcal{N}}(C)$ and $\Gamma^*(G; C)$ as well.

Fundamental limit of seeding via a constant budget. Using the above notations, we establish the following fundamental limit on the impact of seeding in the non-progressive setup, where its proof is given in Section IV-D.

Theorem III.4. *For given graph $G = (V, E)$ and seed budget k, we have*

$$\Delta^*(G) - 2k \leq \min_{C \subset V : |C| \leq k} \Delta^*(G; C) \leq \Delta^*(G).$$

The above theorem implies that just a constant size of seeding does not suffice to achieve an order-wise reduction of diffusion time in the progressive setup. However, it is not the case in the non-progressive setup, as exemplified in what follows: Consider a star graph G, where all nodes except a center node c are just connected to c. In the non-progressive setup, it is easy to check that $\Gamma^*(G) = \Omega(n)$ but $\Gamma^*(G; \{c\}) = 0$, because each "outer" node is only connected to a center node, thus is immediately infected by $+1$ at the next updating chance. Theorem III.4 implies that such a dramatic reduction does not occur in the progressive setup due to such a single seeding, i..e, one has to pay a seed budget scaling to the network size for the purpose.

Finally, we remark that Theorem III.4 is also useful to understand the impact of seeding under the best response diffusion dynamics (i.e., $\beta = \infty$). It is not hard to see that $\Delta^*(G; C) = 0$ provides a sufficient and necessary condition so that every nodes will always select new technology $+1$ eventually under the best response dynamics (in both progressive and non-progressive setups). In particular, when the condition holds with $|C| = O(1)$, the seeding set C is often referred as to *contagion set*, and minimal h for existing such a contagion set is often referred as *contagion threshold* (see [1]). Therefore, Theorem III.4 implies that $\Delta^*(G) = O(1)$ is a necessary condition for the existence of such a contagion set in general graph G, while the author in [1] studied sufficient ones which can be checked for a very limited class of graphs.

IV. Proofs of Theorems

This section provides the proofs of Theorems III.1, III.2, III.3 and III.4. For the notational convenience, we introduce some useful notation. First, for $S \subset V$ and $i \in V \setminus S$, we define $I(S, i)$ as:

$$\begin{aligned}I(S, i) &:= \big[H(S \cup \{i\}) - H(S)\big]_+ \\ &= \big[(1 - h)|N(i)| - 2|N(i) \cap S|\big]_+,\end{aligned}$$

where $[x]_+ = \max\{x, 0\}$ and the last equality is from the definition of $H(\cdot)$ in (6). We note that $I(S, i)$ is the increment of $H(\cdot)$ when a new node i is added to the node set S. In addition, for a linear ordering $\overrightarrow{v} = (v_1, v_2, \ldots, v_n)$, we define

$$\Delta(\overrightarrow{v}) := \max_{1 \leq t \leq n} I(V_{t-1}, v_t). \tag{8}$$

Using the above notation, we simply express the key value $\Delta^*(G)$ as:

$$\Delta^*(G) = \min_{\overrightarrow{v} \in \mathcal{L}(V)} \Delta(\overrightarrow{v}).$$

A. Proof of Theorem III.1

First, we can easily check that the infimum in (5) is achieved when $\boldsymbol{y} = -\mathbf{1}$, i.e.,

$$\tau_{\mathcal{P}} = \inf\{t \geq 0 \mid \mathbb{P}[T(-\mathbf{1}) \geq t] \leq e^{-1}\}, \tag{9}$$

because for any two initial states $\boldsymbol{a}$ with $\boldsymbol{b}$ with $\boldsymbol{a} \geq \boldsymbol{b}$ (component-wise), $T(\boldsymbol{b})$ stochastically dominates $T(\boldsymbol{a})$. To complete the proof of Theorem III.1, it suffices to show that as $\beta \to \infty$,

$$\mathbb{P}\big[T(-\mathbf{1}) > \exp(2\beta\Delta^*(G) + \log\beta)\big] \to 0, \tag{10}$$

$$\mathbb{P}\big[T(-\mathbf{1}) < \exp(2\beta\Delta^*(G) - \log\beta)\big] \to 0. \tag{11}$$

In the rest of this section, we will focus on showing the above convergences.

Given initial state $\boldsymbol{x}(0) = -\mathbf{1}$, let $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n) \in \mathcal{L}(V)$ be a random ordering which corresponds to an ordered sequence of V adopting $+1$. We now state two key lemmas for the proof of (10) and (11), whose proofs are presented in Appendix A and B, respectively.

Lemma IV.1. *For any given ordering $\overrightarrow{v} \in \mathcal{L}(V)$, it follows that as $\beta \to \infty$,*

$$\mathbb{P}\big[T(-\mathbf{1}) > \exp(2\beta\Delta(\overrightarrow{v}) + \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\big] \to 0, \tag{12}$$

$$\mathbb{P}\big[T(-\mathbf{1}) < \exp(2\beta\Delta(\overrightarrow{v}) - \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\big] \to 0. \tag{13}$$

Lemma IV.2. *As $\beta \to \infty$,*

$$\mathbb{P}\big[\Delta(\overrightarrow{\sigma}) = \Delta^*(G)\big] \to 1.$$

Lemma IV.1 gives us the fact that for any diffusion sequence $\overrightarrow{v}$, the bounds of $T(-\mathbf{1})$ are characterized by $\Delta(\overrightarrow{v})$, and Lemma IV.2 states that the diffusion sequence $\overrightarrow{\sigma}$ occurs in the most probably way to minimize Δ, i.e.,

$$\mathbb{P}\left[\overrightarrow{\sigma} \in \arg\min_{\overrightarrow{v} \in \mathcal{L}(V)} \Delta(\overrightarrow{v})\right] \to 1.$$

Now, one can observe that (10) and (11) directly follow from Lemma IV.1 and IV.2. This finishes the proof of Theorem III.1.

B. Proof of Theorem III.2

We first state the following key lemma for the proof of Theorem III.2, which provides the lower and upper bounds of $\Delta^*(G)$ in terms of the degree distribution of graph G.

Lemma IV.3. *For a given graph $G = (V, E)$,*

$$(1-h)d_{\min} \;\leq\; \Delta^*(G) \leq\; (1-h)\; d_{\text{sqr}}$$

where $d_{\min}$ is the minimum degree of G and $d_{\text{sqr}} = \sup\{k = 1, 2, ..., n \mid d_k \geq k\}$ where d_k denotes the k-th largest degree.

The proof of the above lemma is given in Appendix C. One can interpret d_{sqr} as follows: when we sort the nodes in descending order of their degrees, d_{sqr} is the last sorting index which does not exceed the degree of the corresponding node. Clearly,

$$d_{\text{sqr}} \leq d_{\max}, \tag{14}$$

where $d_{\max}$ is the maximum degree of G. Lemma IV.3 is highly useful in the sense that it extremely simplifies the asymptotic analysis of $\Delta^*(G)$, which is a topology-dependent, complex combinatorial value, especially for the graphs simply permitting the analysis on its degree distribution. In the rest of the proof, we let $d(i)$ be the degree of node i.

(a) d-dimensional, random k-regular, and small-world graphs: These graphs have bounded maximum degree, i.e., $d_{\max} = O(1)$. Thus, from Lemma IV.3 and (14), the conclusions of Theorem III.2 for the three graphs immediately follow.

(b) Scale-free graph: From Lemma IV.3, it suffices to show $d_{\text{sqr}} = O(n^{1/\gamma})$ for scale-free graphs with $\gamma > 1$. To this end, let $f(k)$ be the fraction of nodes that have degree k, where $f(k) = \Theta(k^{-\gamma})$ from the definition of the scale-free graphs. Then we have for each $i = 1, ..., n$,

$$d(i) = k \quad \text{if} \quad \sum_{k'=k+1}^{n} f(k') < \frac{i}{n} \leq \sum_{k'=k}^{n} f(k').$$

Remarking that $\int x^{-\gamma} \mathrm{d}x = \frac{x^{1-\gamma}}{1-\gamma}$, the following holds:

$$\sum_{k'=k}^{n} f(k') = \theta(k^{1-\gamma}). \tag{15}$$

Now, to understand d_{sqr} of scale-free graphs, suppose one draws the line $y = x$ and considers a linear interpolation of points $(i, d(i))$ on (x, y)-plane. Then, the x value of the intersecting point is rounded down to the exact value of d_{sqr}. From this observation and (15), it follows that $\frac{d_{\text{sqr}}}{n} = \Theta(d_{\text{sqr}}^{1-\gamma})$, implying $d_{\text{sqr}} = \Theta(n^{1/\gamma})$. Hence, the conclusion of Theorem III.2 for the scale-free graph follows.

(c) ER graph: For ER graph $G = (V, E)$ with $np = \omega(\log n)$, it is elementary to show that as $n \to \infty$,

$$\mathbb{P}\left[|d(i) - (n-1)p| \leq \tfrac{(n-1)p}{2}, \;\; \forall i \in V\right] \to 1,$$

where one can use the Chernoff inequality and the union bound to prove it. This means that $d(i) = \Theta(np)$ for all $i \in V$, thus $d_{\min} = \Theta(np)$ and $d_{\max} = \Theta(np)$ with high probability. Hence, from Lemma IV.3 and (14), the conclusion of Theorem III.2 for the ER graph follows.

C. Proof of Theorem III.3

First, it is clear that Algorithm 1 needs $O(n^2)$ time to terminate since it has n iterations and each iteration takes $O(n)$ time. In what follows, we will focus on the proof that it will output a correct answer. To begin with, we let ordering $\overrightarrow{v} = (v_1, v_2, ..., v_n) \in \mathcal{L}(V)$ be an ordering where node v_t is selected as i^* in the t-th iteration of Algorithm 1. Our goal is to prove that $\overrightarrow{v}$ is a minimizer of function $\Delta(\cdot)$, i.e.,

$$\Delta(\overrightarrow{v}) = \Delta^*(G). \tag{16}$$

To this end, consider an ordering $\overrightarrow{w} \in \mathcal{L}(V)$ and construct an ordering $\overrightarrow{w}^\dagger \in \mathcal{L}(V)$ by moving v_1 to the first place on ordering $\overrightarrow{w}$, i.e,.

$$\overrightarrow{w}^\dagger = (v_1, w_1, ..., w_{s-1}, w_{s+1}, ..., w_n)$$

where $w_s = v_1$. We will show that

$$\Delta(\overrightarrow{w}) \geq \Delta(\overrightarrow{w}^{\dagger}). \tag{17}$$

The above inequality suffices to show (16) since one can also recursively move v_l to the l-th place for $\Delta(\overrightarrow{w}) \geq \Delta(\overrightarrow{v})$.

For node $i \in V$, let $W(i)$ and $W^{\dagger}(i)$ denote the sets of nodes before node i in orderings $\overrightarrow{w}$ and $\overrightarrow{w}^{\dagger}$, respectively. We have $W(i) \subset W^{\dagger}(i)$ for all node i except v_1 due to our construction of $\overrightarrow{w}^{\dagger}$. Hence, form the monotonicity of $I(S,i)$ with respect to S, it follows that

$$I(W(i),i) \geq I(W^{\dagger}(i),i) \quad \text{for all } i \neq v_1. \tag{18}$$

Furthermore, we observe that

$$\max_{i \in V} I(W(i),i) \geq I(W(w_1),w_1) \geq I(W^{\dagger}(v_1),v_1),$$

where the second inequality is because Algorithm 1 selects v_1 at its first iteration. Combining the above inequality with (18), we conclude that

$$\max_{i \in V} I(W(i),i) \geq \max_{i \in V} I(W^{\dagger}(i),i),$$

which implies (17) and completes the proof of Theorem III.3.

D. Proof of Theorem III.4

Consider an arbitrary seed set $C \subset V$ and node $i \in C$. It is enough to show that

$$\Delta^*(G) \;\geq\; \Delta^*(G;C) \;\geq\; \Delta^*(G;C \setminus \{i\}) - 2, \tag{19}$$

where the first inequality is trivial.

For notational simplicity, we let $D = C \setminus \{i\}$ and $m = |V \setminus D|$. We construct an ordering $\overrightarrow{w} \in \mathcal{L}(V \setminus D)$ of length m as follows:

$$\overrightarrow{w} = (w_1, w_2, ..., w_{m-1}, w_m) = (v_1, v_2, ...,\; v_{m-1},\; i),$$

where $\overrightarrow{v} = (v_1, ..., v_{m-1}) \in \mathcal{L}(V \setminus C)$ of length $(m-1)$ is an ordering which the minimization in $\Delta^*(G;C)$ finds, i.e.,

$$\Delta^*(G;C) \quad = \quad \max_{1 \leq t \leq m-1} I(C \cup V_{t-1}, v_t).$$

Then, we have $W_t = V_t$ for $t = 1, ..., m-1$, where $V_t = \{v_1, ..., v_t\}$ and $W_t = \{w_1, ..., w_t\}$. Thus the construction of $\overrightarrow{w}$ allows to have that for all $t = 1, ..., m-1$,

$$\begin{aligned}
&I(C \cup V_{t-1}, v_t) \\
&= \big[(1-h)|N(w_t)| - 2|N(w_t) \cap (C \cup W_{t-1})|\big]_+ \\
&\overset{\text{(a)}}{\geq} \big[(1-h)|N(w_t)| - 2|N(w_t) \cap (D \cup W_{t-1})|\big]_+ - 2 \\
&= I(D \cup W_{t-1}, w_t) - 2,
\end{aligned} \tag{20}$$

where, the inequality (a) holds because node w_t can have at most one edge to node i. Furthermore, one can observe that

$$I(D \cup W_{m-1}, w_m) = I(V \setminus \{i\}, i) = 0 \tag{21}$$

since $(1-h)|N(i)| - 2|N(i)| < 0$. Using (20) and (21), one can conclude that

$$\begin{aligned}
\Delta^*(G;C) &= \max_{1 \leq t \leq m-1} I(C \cup V_{t-1}, v_t) \\
&\geq \max_{1 \leq t \leq m} I(D \cup W_{t-1}, w_t) - 2 \geq \Delta^*(G;D) - 2
\end{aligned}$$

where the last inequality holds from the definition of $\Delta^*(G;D)$. This shows the second inequality of (19) and completes the proof of Theorem III.4.

V. Experimental Results and Seeding Algorithm

In this section, we provide numerical evaluations that support our analytical results. First, we demonstrate via simulations that estimating diffusion times via Δ^* computation (i.e., Algorithm 1) is good under a variety of scenarios. Second, we propose a simple seeding algorithm in the progressive setup inspired by minimizing Δ^* greedily, and verify that it outperforms other natural heuristics. We use the data set extracted by the authors in [31], depicted in Figure 1(a). These data set consists of an undirected graph consisting of 4039 nodes and 88234 edges where each node corresponds to a Facebook user and a pair of nodes has an edge if their corresponding users are in each other's *Friend Lists* of Facebook.

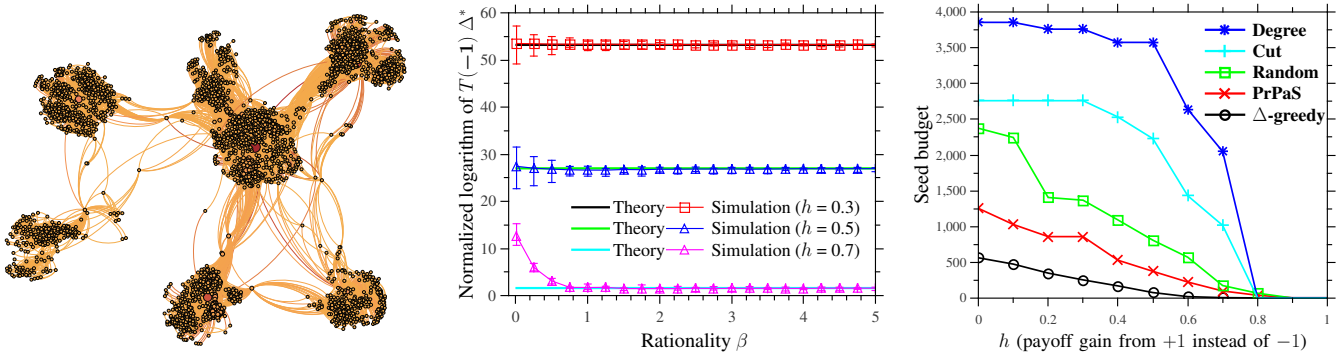
(a) Verification of Δ^*: For varying values of β and h, Δ^* computed by Algorithm 1 is compared with the value from simulations which are obtained by the following way. For given β and h, we run 1000 random simulations and obtain their hitting times. From those hitting time samples, we calculate the simulated Δ^* by taking the logarithm of the median of hitting time samples and dividing it by 2β (see the formula in Theorem III.1). Figure 1(b) plots our numerical results, where we observe that for three choices of $h = 0.3, 0.5, 0.7$, our analytic and simulated Δ^*'s are almost same even for small β, although Theorem III.1 requires $\beta \to \infty$. These experimental results indicate that our characterization of the diffusion time is highly correct even when people behave with some degree of irrationality (i.e., small β).

(b) Greedy seeding algorithm: We propose the following greedy seeding algorithm using the direct computations of Δ^* (i.e., Algorithm 1), where we call it **Δ-greedy**. We assume the seed budget k (i.e., it can selects k nodes initially forcing them to have +1 strategy) and the algorithm description is as follows:

- **Δ-greedy.** It iteratively chooses set of k nodes, denoted by $c_1, ..., c_k$. Let $C_t = \{c_1, ..., c_t\}$ be the intermediate set containing the first t seeds. Then, c_{t+1} is selected as a maximizer i of $I(C_{t+1}, i)$ among minimizers of $\Delta^*(G; C_t \cup \{i\})$, e.g., if the minimizer i of $\Delta^*(G; C_t \cup \{i\})$ is unique, then it becomes c_{t+1}. One can choose any minimizer of $\Delta^*(G; C_t \cup \{i\})$, but we propose to choose one which additionally maximizes $I(C_{t+1}, i)$ for more reduction in Δ^* in $c_{t+2}, c_{t+3}, \ldots$.

We compare **Δ-greedy** to the following heuristics:

- **Degree.** This chooses k nodes in the order of their degrees.
- **Random.** This selects k nodes uniformly at random.
- **PrPaS.** This is a heuristic proposed in [5] under the non-progressive assumption. The algorithm first partitions the graph into clusters and randomly selects seeds in each cluster of which budget is allocated proportionally to its size.
- **Cut.** This runs k iterations where at each iteration a node with the maximum number of edges is selected, and then removed from the seed candidates.



(a) Each of 4039 circles correspond to a Facebook user and 88234 edges connect them based on FriendLists in Facebook.

(b) $\Delta^*(G)$ from Algorithm 1 and simulations for $h = 0.3, 0.5, 0.7$ as β varies. Error bars indicate the interval including all samples.

(c) Required budget for $\Delta^* < 1$ with $\beta = 5$.

Fig. 1. The Facebook network and simulation results.

Figure 1(c) shows the required seed budgets for making $\Delta^* < 1$ (i.e., fast diffusion) under the above five algorithms, under various values of h and $\beta = 5$. Recall that every point is the average over 1000 samples. We observe that **Δ-greedy** is superior to other four algorithms, which demonstrates the value of polynomial computation of Δ^* by Algorithm 1 for the seeding problem. Moreover, the proposed **Δ-greedy** might be an optimal seeding algorithm in the progressive setup, where its theoretical analysis is an interesting open question. Such a seeding algorithm is impossible in the non-progressive setup since Γ^* is computationally intractable.

VI. Conclusion

In this paper, we have studied how the diffusion speed of new technology under a noisy game-based model and the progressive assumption. We have characterized the diffusion time via the combinatorial value which can be computed by a polynomial time algorithm and also provided the asymptotic analysis of the diffusion time for a group of popular graphs. Finally, we have studied the fundamental limit of seeding in the progressive setup, and show that the diffusion speed is impossible to significantly accelerate with just a small-budget. Interestingly enough, all these results are in part contrast to those in the non-progressive setup. We believe that the computational tools developed in this paper will be of broader interest to tackle other related problems in this area.

References

[1] S. Morris, "Contagion," *The Review of Economic Studies*, vol. 67, no. 1, pp. 57–78, 2000.

[2] G. Ellison, "Learning, local action, and coordination," *Econometrica*, vol. 61, no. 5, pp. 1047–1071, Sep. 1993.

[3] N. Immorlica, J. Kleinberg, M. Mahidian, and T. Wexler, "The role of compatibility in the diffusion of technologies through social networks," in *Proc. of ACM EC*, 2007.

[4] A. Montanari and A. Saberi, "Convergence to equilibrium in local interaction games," in *Proc. of IEEE FOCS*, 2009.

[5] J. Ok, Y. Jin, J. Shin, and Y. Yi, "On maximizing diffusion speed in social networks: impact of random seeding and clustering," in *Proc. of ACM SIGMETRICS*, 2014.

[6] L. E. Blume, "The statistical mechanics of strategic interaction," *Games and Economic Behavior*, vol. 5, no. 3, pp. 387–424, 1993.

[7] H. P. Young, *Individual strategy and social structure: An evolutionary theory of institutions*. Princeton University Press, 1998.

[8] M. Kandori, G. J. Mailath, and R. Rob, "Learning, mutation, and long run equilibria in games," *Econometrica*, vol. 61, no. 1, pp. 29–56, Jan. 1993.

[9] Y. Jin, J. Ok, Y. Yi, and J. Shin, "On the impact of global information on diffusion of innovations over social networks," in *Proc. of IEEE Infocom NetSci*, 2013.

[10] H. R. H.Arkes and C. Blumer, "The psychology of sunk cost," *Organizational behavior and human decision processes*, vol. 35, no. 1, pp. 124–140, 1985.

[11] P. Domingos and M. Richardson, "Mining the network value of customers," in *Proc. of ACM SIGKDD*, 2001.

[12] P. Netrapalli and S. Sanghavi, "Learning the graph of epidemic cascades," in *Proc. of ACM SIGMETRICS*, 2012.

[13] M. Lelarge, "Coordination in network security games," in *Proc. of INFOCOM*, 2012.

[14] E. Coupechoux and M. Lelarge, "Impact of clustering on diffusions and contagions in random networks," in *Proc. of IEEE NetGCooP*, 2011.

[15] M. Lelarge, "Diffusion and cascading behavior in random networks," *Games and Economic Behavior*, vol. 75, no. 2, pp. 752–775, 2012.

[16] S. Liu, L. Ying, and S. Shakkottai, "Influence maximization in social networks: An ising-model-based approach," in *Proc. of IEEE Allerton*, 2010.

[17] W. O. Kermack and A. G. McKendrick, "Contributions to the mathematical theory of epidemics. ii. the problem of endemicity," in *Proc. of the Royal society of London. Series A*, 1932.

[18] E. Ising, "Beitrag zur theorie des ferromagnetismus," *Zeitschrift für Physik A Hadrons and Nuclei*, vol. 31, no. 1, pp. 253–258, 1925.

[19] N. T. J. Bailey, *The Mathematical Theory of Infectious Diseases and Its Applications*. Hafner Press, 1975.

[20] R. M. Anderson and R. M. May, *Infectious Diseases of Humans*. Oxford University Press, 1991.

[21] H. W. Hethcote, "The mathematics of infectious diseases," *SIAM review*, vol. 42, no. 4, pp. 599–653, 2000.

[22] D. Kempe, J. Kleinberg, and E. Tardos, "Maximizing the spread of influence through a social network," in *Proc. of ACM SIGKDD*, 2003.

[23] M. Draief and A. Ganesh, "A random walk model for infection on graphs: spread of epidemics rumours with mobile agents," *Discrete Event Dynamic Systems*, vol. 21, no. 1, pp. 41–61, Mar. 2011.

[24] S. Banerjee, A. Gopalan, A. K. Das, and S. Shakkottai, "Epidemic spreading with external agents," *arXiv preprint arXiv:1206.3599*, 2012.

[25] A. Ganesh, L. Massouli, and D. Towsley, "The effect of network topology on the spread of epidemics," in *Proc. of IEEE Infocom*, 2003.

[26] D. Chakrabarti, Y. Wang, C. Wang, J. Leskovec, and C. Faloutsos, "Epidemic thresholds in real networks," *ACM Transactions on Infroamtion and System Security*, vol. 10, no. 4, pp. 13:1–13:25, 2005.

[27] D. Kempe, J. Kleinberg, and E. Tardos, "Influential nodes in a diffusion model for social networks," in *Proc. of Intl. Colloq. on Automata, Languages and Programming*. Springer, 2005, pp. 1127–1138.

[28] D. McFadden, *Chapter 4: Conditional logit analysis of qualitative choice behavior in Frontiers in Econometrics*. Academic Press, New York, 1973.

[29] D. Mookherjee and B. Sopher, "Learning behavior in an experimental matching pennies game," *Games and Economic Behavior*, vol. 7, no. 1, pp. 62–91, 1994.

[30] R. D. McKelvey and T. R. Palfrey, "Quantal response equilibria for normal form games," *Games and Economic Behavior*, vol. 10, no. 1, pp. 6–38, 1995.

[31] J. McAuley and J. Leskovec, "Learning to discover social circles in ego networks," in *Proc. of NIPS*, 2012.

APPENDIX

A. Proof of Lemma IV.1

In this section, we present the proof of Lemma IV.1. To begin with, we study random variable $T(-\mathbf{1})$ conditioned on the event that the progressive diffusion occurs along ordering $\overrightarrow{v} = (v_1, \ldots, v_n)$, i.e., $\overrightarrow{\sigma} = \overrightarrow{v}$. Then the hitting time $T(-\mathbf{1})$ can be written as a summation of random variables as follow:

$$T(-\mathbf{1}) = \sum_{t=1}^{n} T(V_{t-1}, v_t)$$

where $V_t = \{v_1, ..., v_t\}$ and for subset $S \subset V$ and node $i \in V \setminus S$, we define $T(S, i)$ be the time duration till node i selects $+1$, while nodes in S and $V \setminus (S \cup \{i\})$ remain to choose $+1$ and -1, respectively. Since node i has an exponential clock of unit rate to update its strategy, $T(S, i)$ is an exponential random variable with rate parameter $\lambda(S, i)$, where $\lambda(S, i)$ is the probability that node i selects $+1$ when its clock ticks. Furthermore, from (4), one can check that

$$\tfrac{1}{2}\exp(-2\beta I(S, i)) \le \lambda(S, i) \le \exp(-2\beta I(S, i)). \quad (22)$$

Proof of (12)**.** From (22), we have that for all $t = 1, ..., n$,

$$\tfrac{1}{2}\exp(-2\beta\Delta(\overrightarrow{v})) \le \lambda(V_{t-1}, v_t), \quad (23)$$

where we remind that $\Delta(\overrightarrow{v}) = \max_{1\le t\le n} I(V_{t-1}, v_t)$. This implies that $T(V_{t-1}, v_t)$ for all t is stochastically dominated by an exponential random variable with parameter $\frac{1}{2}\exp(-2\beta\Delta(\overrightarrow{v}))$. Using this, we have

$$\begin{aligned}
&\mathbb{P}\left[T(-\mathbf{1}) > \exp(2\beta\Delta(\overrightarrow{v}) + \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\right] \\
&\le \mathbb{P}\left[Z > \exp(2\beta\Delta(\overrightarrow{v}) + \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\right] \\
&= \sum_{k=0}^{n-1} \frac{e^{-\frac{\beta}{2}}(\frac{\beta}{2})^k}{k!} \;\le\; ne^{-\frac{\beta}{2}}(\tfrac{\beta}{2})^n \;\xrightarrow[\beta\to\infty]{}\; 0,
\end{aligned}$$

where Z is the summation of n independent exponential random variables with parameter $\frac{1}{2}\exp(-2\beta\Delta(\overrightarrow{v}))$. This completes the proof of (12).

Proof of (13)**.** Now we define t^* as

$$t^* \in \arg\max_{1\le t\le n} I(V_{t-1}, v_t).$$

Using this notation, we derive that

$$\begin{aligned}
&\mathbb{P}\left[T(-\mathbf{1}) < \exp(2\beta\Delta(\overrightarrow{v}) - \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\right] \\
&\le \mathbb{P}\left[T(V_{t^*-1}, v_{t^*}) < \exp(2\beta\Delta(\overrightarrow{v}) - \log\beta) \mid \overrightarrow{\sigma} = \overrightarrow{v}\right] \\
&\overset{(a)}{\le} 1 - e^{-\frac{1}{\beta}} \;\le\; \frac{1}{\beta} \;\xrightarrow[\beta\to\infty]{}\; 0,
\end{aligned}$$

where the inequality (a) from the definition of t^* and (22). This completes the proof of (13).

B. Proof of Lemma IV.2

Consider an ordering $\overrightarrow{v} = (v_1, \ldots, v_n)$ such that $\Delta(\overrightarrow{v}) \neq \Delta^*(G)$. Then, there exist indices a and b such that $1 \le a < b \le n$ and

$$I(V_{a-1}, v_a) > I(V_{a-1}, v_b) \quad (24)$$

where $V_t = \{v_1, ..., v_t\}$. For the event $\overrightarrow{\sigma} = \overrightarrow{v}$ to occur, node v_a should select $+1$ before node v_b does. Under this observation, we have

$$\begin{aligned}
\mathbb{P}[\overrightarrow{\sigma} = \overrightarrow{v}] \;&\le\; \mathbb{P}[T(V_{a-1}, v_a) < T(V_{a-1}, v_b)] \\
&\le \frac{\exp(-2\beta I(V_{a-1}, v_a))}{\exp(-2\beta I(V_{a-1}, v_a)) + \frac{1}{2}\exp(-2\beta I(V_{a-1}, v_b))}
\end{aligned}$$

where the last inequality is from (22). In the above, the last term goes to 0 as $\beta \to \infty$ due to (24), which implies that

$$\mathbb{P}[\Delta(\overrightarrow{\sigma}) \neq \Delta^*(G)] \;=\; \sum_{\Delta(\overrightarrow{v})\neq\Delta^*(G)} \mathbb{P}[\overrightarrow{\sigma} = \overrightarrow{v}] \;\xrightarrow[\beta\to\infty]{}\; 0.$$

This completes the proof of Lemma IV.2.

C. Proof of Lemma IV.3

We first obtain the lower bound of $\Delta^*(G)$ as follow:

$$\begin{aligned}
\Delta^*(G) &= \min_{\overrightarrow{v}\in\mathcal{L}(V)} \max_{1\le t\le n} I(V_{t-1}, v_t) \ge \min_{\overrightarrow{v}\in\mathcal{L}(V)} I(V_0, v_1) \\
&= \min_{i\in V} I(\emptyset, i) = \min_{i\in V}(1-h)|N(i)| = (1-h)d_{\min}.
\end{aligned}$$

Now we focus on the proof of the upper bound of $\Delta^*(G)$ in Lemma IV.3. Consider an ordering $\overrightarrow{w} = (w_1, \ldots, w_n)$ in increasing order of degree, i.e., $|N(w_t)| \ge |N(w_{t'})|$ if $t \ge t'$ so that we have for all $t = 1, ..., n$

$$d_{n-t+1} = |N(w_t)| \quad (25)$$

where we recall that d_i denotes the i-th largest degree in nodes of graph G. Then, we have $N(w_t) \cap W_{t-1} = N(w_t) \setminus (V \setminus W_{t-1})$ where $W_t = \{w_1, ..., w_t\}$. This implies that

$$\begin{aligned}
|N(w_t)| \ge |N(w_t) \cap W_{t-1}| &\ge |N(w_t)| - |V \setminus W_{t-1}| \\
&= |N(w_t)| - (n - t + 1). \quad (26)
\end{aligned}$$

Using (25) and (26), it follows that

$$\begin{aligned}
\Delta^*(G) \;&=\; \min_{\overrightarrow{v}\in\mathcal{L}(V)} \max_{1\le t\le n} I(V_{t-1}, v_t) \\
&\le \max_{1\le t\le n} I(W_{t-1}, w_t) \\
&= \max_{1\le t\le n} \left[(1-h)|N(w_t)| - 2|N(w_t) \cap W_{t-1}|\right] \\
&\overset{(b)}{\le} \max_{1\le t\le n} \left[(1-h)|N(w_t)| - 2[|N(w_t)| - (n-t)\right]_+ \\
&\overset{(c)}{=} \max_{1\le i\le n} \left[(1-h)d_i - 2[d_i - i]_+\right], \quad (27)
\end{aligned}$$

where (b) is due to (26) and for (c) we use (25) by replacing $(n-t+1)$ with i. Moreover, by the definition of d_{sqr}, one can observe that $[d_i - i]_+ = d_i - i \ge 0$ for all $i \le d_{\text{sqr}}$ and

$$\begin{aligned}
&\max_{1\le i\le d_{\text{sqr}}} \left[(1-h)d_i - 2[d_i - i]_+\right] \\
&= \max_{1\le i\le d_{\text{sqr}}} \left[2i - (1+h)d_i\right] \\
&\le \max_{1\le i\le d_{\text{sqr}}} (1-h)i = (1-h)d_{\text{sqr}} \quad (28)
\end{aligned}$$

where the last inequality is due to $d_i \ge i$ for all $i \le d_{\text{sqr}}$ and $0 \le h \le 1$. In addition, we obtain

$$\begin{aligned}
\max_{d_{\text{sqr}}<i\le n} \left[(1-h)d_i - 2[d_i - i]_+\right] &= \max_{d_{\text{sqr}}<i\le n} (1-h)d_i \\
&\le (1-h)d_{\text{sqr}}, \quad (29)
\end{aligned}$$

where the first equality is from the fact that $[d_i - i]_+ = 0$ for all $i > d_{\text{sqr}}$ and the last inequality holds since d_i is decreasing with respect to i (i.e., the d_{sqr}-th or later degree is less than d_{sqr}). Finally, by combining (27), (28) and (29), one can conclude that

$$\Delta^*(G) \;\le\; \max_{1\le i\le n} \left[(1-h)d_i - 2[d_i - i]_+\right] \;\le\; (1-h)d_{\text{sqr}},$$

and this completes the proof of Lemma IV.3.