Delay-Capacity Tradeoffs for Mobile Networks with Lévy Walks and Lévy Flights

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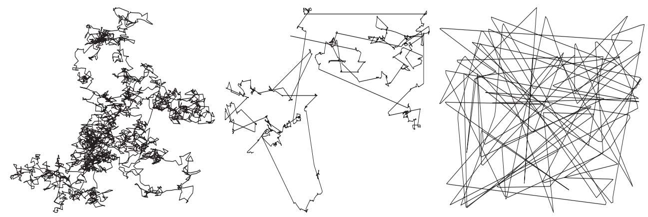
***Abstract*—This paper analytically derives the delay-capacity tradeoffs for Lévy mobility: Lévy walks and Lévy flights. Lévy mobility is a random walk with a power-law flight distribution. α is the power-law slope of the distribution and $0 < \alpha \leq 2$. While in Lévy flight, each flight takes a constant flight time, in Lévy walk, it has a constant velocity which incurs strong spatio-temporal correlation as flight time depends on traveling distance. Lévy mobility is of special interest because it is known that Lévy mobility and human mobility share several common features including heavy-tail flight distributions. Humans highly influence the mobility of nodes (smartphones and cars) in real mobile networks as they carry or drive mobile nodes. Understanding the fundamental delay-capacity tradeoffs of Lévy mobility provides important insight into understanding the performance of real mobile networks. However, its power-law nature and strong spatio-temporal correlation make the scaling analysis non-trivial. This is in contrast to other random mobility models including Brownian motion, random waypoint and i.i.d. mobility which are amenable for a Markovian analysis. By exploiting the asymptotic characterization of the joint spatio-temporal probability density functions of Lévy models, the order of critical delay, the minimum delay required to achieve more throughput than $\Theta(1/\sqrt{n})$ where n is the number of nodes in the network, is obtained. The results indicate that in Lévy walk, there is a phase transition that for $0 < \alpha < 1$, the critical delay is constantly $\Theta(n^{1/2})$ and for $1 \leq \alpha \leq 2$, is $\Theta(n^{\alpha/2})$. In contrast, Lévy flight has critical delay $\Theta(n^{\alpha/2})$ for $0 < \alpha \leq 2$.**

I. Introduction

Since the seminal work by Gupta and Kumar [1] on the capacity of wireless networks, delay and throughput tradeoffs of wireless networks have been extensively studied under various mathematical techniques, scheduling algorithms, channel models, mobility models and physical layer techniques. Among them, arguably the most notable contribution is the work by Grossglauser and Tse [2] showing that per-node throughput remains constant ($\Theta(1)$) when node mobility is used for communication. This result is surprising because Gupta and Kumar [1] showed that per-node throughput ($O(1/\sqrt{n})$) in wireless networks with no mobility diminishes as the number of nodes n increases. This throughput gain is achieved at the cost of larger delays.

The amount of delay that a network needs to sacrifice to guarantee a given throughput has been studied under various mobility models [3]–[6]. In particular, Sharma et al.[7] studied the minimum delays required to achieve more throughput than $O(1/\sqrt{n})$ under various mobility models including i.i.d., random waypoint (RWP), random direction (RD) and Brownian motion (BM). This minimum delay is called *critical delay*. However, although the work is of high value in terms of providing a framework for studying delay-capacity scaling for wireless networks under a family of random walk models, the practical values of these models are uncertain. While these models are simple enough for mathematical tractability, they do not reflect realistic mobility patterns commonly exhibited in real mobile networks.



(a) BM (b) Lévy walk (c) RWP

Fig. 1. Sample trajectories of (a) BM, (b) Lévy walk and (c) RWP.

Humans are a big factor in mobile networks as most mobile nodes or devices (smartphones and cars) are carried or driven by humans. Recent studies [8]–[10] on human mobility show that flight length distributions have a heavy-tail tendency where *flights* are defined to be the longest straight line trip of an object (e.g., particles or humans) from one location to another without a directional change or pause. These mobility patterns are well-modeled by Lévy process [11].

Lévy mobility is a random walk mobility with a power-law flight distribution, $1/z^{1+\alpha}$ where z is a flight length and $0 < \alpha \leq 2$. It also represents a random walk mobility with just a heavy-tail flight distribution [11], [12]. Intuitively, such a random walk contains many short flights and a small yet significant number of exceptionally long flights. With different values of α, the flight patterns of Lévy mobility models are widely different. Smaller α induces a larger number of long flights. This type of mobility patterns is significantly different from Brownian motion and RWP as illustrated in Fig. 1. In the literature, there are two types of Lévy mobility models: *Lévy flight* (LF) and *Lévy walk* (LW). In Lévy flight, every flight takes a *constant time* irrespective of its flight length and in Lévy walk, it takes a *constant velocity*.

Unfortunately, understanding tradeoffs between throughput and delay under Lévy mobility is technically very challenging and underexplored. Unlike the other random walk models permitting mathematical tractability, Lévy process is not very well understood mathematically despite significant studies on

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Lévy process in mathematics and physics. Thus, the conventional techniques [6], [7] used to study delay-capacity tradeoffs cannot be applied to Lévy models, especially to Lévy walk which has high spatio-temporal correlation. While Lévy walk defies the discretization process required for Markovian analysis, its mathematical characteristics of continuous Lévy models such as the closed-form formulas for joint spatio-temporal probability density function (PDF) are practically unknown.

Our main contribution is to analytically derive the tradeoffs between delay and capacity for both Lévy models. Our technique is unique in that we use the asymptotic characterization of joint spatio-temporal PDF and diffusion equation of Lévy models without solving their closed forms. As different α induces different mobility patterns, it also induces different delay-capacity tradeoffs. Below we summarize our main results.

Mobility	α	Critical Delay
Lévy walk	$\alpha \in (0, 1)$	$\Theta(\sqrt{n})$
	$\alpha \in [1, 2]$	$\Theta(n^{\alpha/2})$
Lévy flight	$\alpha \in (0, 2]$	$\Theta(n^{\alpha/2})$

Given that many human mobility traces are shown to have α between 0.53 and 1.81 [8], according to our results, mobile networks assisted by human mobility have critical delays between $\Theta(n^{0.27})$ and $\Theta(n^{0.91})$. Note that our results give much more detailed prediction of critical delay for such mobile networks depending on α while BM and RWP always show $\Theta(n)$ and $\Theta(n^{0.5})$ for their critical delays [7].

The rest of the paper is organized as follows. We introduce our system model in Section III and the Lévy mobility model parameterized with α in Section IV and study critical delay of Lévy-walk mobility in Section VI based on the preliminary given in Section V. Finally, we provide a high level interpretation and concluding remarks in Section VII.

II. Related Work

Gupta and Kumar [1] showed that the per-node capacity of random wireless networks with n static nodes scales as a function of $O(1/\sqrt{n})$ and proposed a scheme achieving $\Theta(1/\sqrt{n \log n})$. The result is later enhanced to $\Theta(1/\sqrt{n})$ by exercising individual power control [13], [14]. Grossglauser and Tse [2] made a breakthrough by proving that a constant per-node throughput is achievable by using mobility when the nodes follow ergodic and stationary mobility models. This disproves the conventional belief that node mobility can negatively impact network capacity as it causes connectivity breakup and channel quality degradation. It is later shown that the gain comes at the cost of larger delay [5], [15].

Many follow-up studies[3]–[5], [15]–[19] have been devoted to understand, characterize and exploit the tradeoffs between throughput and delay. Especially, the delay required to obtain the constant throughput $\Theta(1)$ has been later studied under various mobility models [4], [18]–[21]. The key message is that the delay of 2-hop relaying proposed in [2] is $\Theta(n)$ for most mobility models such as i.i.d. mobility, random direction, random waypoint and Brownian motion models. An important question is how much delay needs to be increased to achieve asymptotically higher throughput than $\Theta(1/\sqrt{n})$. This has been studied under the notion of critical delay [6], [7] for two families of random mobility models: *hybrid random walk* and *random direction*. Hybrid random walk splits the network of size 1 with $n^{2\beta}$ cells and further splits a cell into $n^{1-2\beta}$ subcells. Then, a node moves to a random subcell of an adjacent cell in every unit time slot. In this model, i.i.d. mobility corresponds to $\beta = 0$ and random walk mobility corresponds to $\beta = 1/2$. For any β, critical delay is proved to be $\Theta(n^{2\beta})$. Random direction chooses a random direction within $[0, 2\pi]$ and moves to the selected direction with a distance of $n^{-\gamma}$ with a velocity $n^{-1/2}$. In this model, random waypoint (or random direction) and Brownian motion are represented with $\gamma = 0$ and $\gamma = 1/2$, respectively. The critical delay is proved as $\Theta(n^{1/2+\gamma})$.

III. Model Description

A. System Model

We consider a wireless mobile network indexed by n, where in the n-th network, n nodes are distributed uniformly on a completely wrapped-around square S whose width and height scale as $\sqrt{n}$ and density is fixed to 1 with increasing n.[1] We assume all nodes are homogeneous in that each node generates data with the same intensity to a per-source destination. The packet generation process at each node is assumed to be independent of node mobility.

The source-to-destination packet delivery can be delivered by either direct one-hop transmission or over multi-hops, say k hops, using relay nodes. We call it k-hop relay transmission. We assume that none of relay nodes generates packets.

To model interference in wireless networks, we use the protocol model as in [1], [21], under which nodes transmit packet successfully at a constant rate W bits/sec, if and only if the following is met: for a transmitter i, a receiver j and every other node $k \neq i, j$ transmitting currently,

$$d(\vec{X}^k(t), \vec{X}^j(t)) \geq (1 + \Delta) d(\vec{X}^i(t), \vec{X}^j(t)), \quad \text{for } \Delta > 0,$$

where $\vec{X}^i(t)\,(\in \mathbb{R}^2)$ denotes node i's location at time t and $d(x, y)$ denotes distance between two locations x, y.

A packet can be delivered through a scheduling scheme which consists of *replication* or *forwarding*. We assume that only source nodes replicate packets and all other relay nodes forward them. As the names imply, replication copies a packet and the packet transmitter keeps the packet, whereas in forwarding the transmitter does not keep the original packet after successful transmission. This selective replication and forwarding depending on the node type are often applied to suppress the overflow of redundant packets in the network. Packets are delivered in two ways: *neighbor capture* and *multi-hop capture*. In the neighbor capture, using mobility, relay or source nodes are located within the communication range of the destination. In multi-hop capture, a source establishes a multi-hop path to the destination and delivers the packets over the path. We assume a fluid packet model [21] so that the delivery can occur immediately even in the case of multi-hop

[1] This model is often referred to as an extended model. In another model, called a unit network model, the network area is fixed to 1 and density increases as n while the spacing and velocity of nodes scale as $1/\sqrt{n}$.

capture because the transmission delay is negligible compared to the delay from node mobility. We denote by Π the class of all scheduling schemes conforming the descriptions above.

B. Performance Metrics

The primary performance metric in many networking systems is throughput measured by the long-term average of received packets aggregated over nodes, defined as:

Definition 1 (Throughput): For a scheduling scheme π, the throughput λ_π is:

$$\lambda_\pi \triangleq \liminf_{t\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_\pi^i(t)}{t},$$

where $\lambda_\pi^i(t)$ is the total number of bits received at a destination node i up to time t.[2]

Another important metric is delay. Let $D_\pi^{i,j}$ be the individual packet delay that a packet j experiences to arrive at a destination node i from its source under a scheduling scheme π. The average delay of a scheme π is defined as:

Definition 2 (Average delay):

$$D_\pi \triangleq \lim_{k\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{k}\sum_{j=1}^{k} D_\pi^{i,j}.$$

We give special attention to the notion of critical delay C_Π, first introduced in [7] and defined as:

Definition 3 (Critical delay): C_Π is the minimum average delay that must be tolerated under a given mobility model to achieve a per-node throughput of $\omega(1/\sqrt{n})$, i.e.,

$$C_\Pi \triangleq \inf_{\{\pi\in\Pi:\lambda_\pi=\omega(1/\sqrt{n})\}} D_\pi.$$

Per-node throughput $\Theta(1/\sqrt{n})$ is achievable by a scheduling scheme in static multi-hop networks [1]. Since node mobility can increase throughput at the cost of larger delay, critical delay quantifies the amount of delay that a network should sacrifice to achieve the guaranteed "baseline" throughput. It can be used as a simple, yet useful metric for a mobility model, representing how sensitive the delay is to increase per-node throughput.

IV. Lévy Mobility Model

In this section, we formally define *Lévy mobility model*, and explain the technical challenges that preclude the use of the conventional techniques to our model, requiring us to take a different approach to study critical delay.

A. Lévy Walk vs. Lévy Flight

Lévy walk and Lévy flight are separately treated in many literatures [22]–[24]. Lévy flight takes a *constant time* for every flight irrespective of the flight length of a flight whereas Lévy walk takes a *constant velocity* for each flight. Thus, in Lévy walk, it takes a flight time proportional to the flight length. The distinction between Lévy walk and flight is often made with their mobility speed. Lévy flight is a "fast" mobility model in that the time taken for movement is comparable to the packet transmission time in the multi-hop network. In a similar context, Lévy walk falls into a "slow" mobility model. An experimental velocity model suggested as a function of flight length in [8] verifies that a human mobility lies in between Lévy walk and Lévy flight. For convenience, we use Lévy mobility model to indicate both of Lévy walk and Lévy flight, unless explicitly stated.

Lévy mobility follows a Lévy distribution, expressed by the Fourier transformation for its flight length Z (moving distance of a single random walk), and its PDF is given by:

$$f_{Z,\alpha}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itz-|Ct|^\alpha}dt,$$

where C is a scale factor and α is a distribution parameter. α ranges over $(0,2]$ and determines the flight length distribution. Lévy distribution for $0<\alpha<1$ has infinite variance and mean, while the distribution for $1\le\alpha<2$ has infinite variance but finite mean. Lévy mobility for $\alpha=2$ is Brownian motion and has finite variance and mean. Due to the complex form of the distribution, Lévy distribution is often given as a power-law type of asymptotic form, closely approximating the tail part of the distribution:

$$f_{Z,\alpha}(z) \sim \frac{1}{z^{1+\alpha}}.$$

We assume that the flight length Z has a lower bound at 1[3] and no upper bound irrespective of network size which is proportional to $\sqrt{n}$.

B. Challenges

There are two general techniques in studying critical delay for random walk mobility models. One is to discretize mobility and then apply a Markovian analysis [7], and the other is to use continuous mobility models and solve directly the diffusion equation to obtain joint spatio-temporal PDF [6]. Unfortunately, both techniques cannot be applied directly to Lévy models. While the discretization of mobility models may be applied to Lévy flight, the same cannot be applied to Lévy walk because of high spatio-temporal correlation of Lévy walk. While in Lévy flight, a node moves to its next destination in a unit time (or a constant time), in Lévy walk, its travel time depends on the distance to the next destination. This joint spatio-temporal coupling makes the future motion of a Lévy walker dependent on its past history. Thus, solving directly the diffusion equation of Lévy walk is very challenging and no mathematical solutions are yet available. Our approach is to derive critical delay from an asymptotic characterization of the joint spatio-temporal PDF without the exact solution.

V. Preliminaries for Critical Delay

Computing critical delay consists of multiple steps. We start by following the initial step in [6], [7] which connects critical delay to the first exit time. Critical delay can simply be regarded as *the maximum time duration that a node cannot exit from a disc of a radius $\Theta(\sqrt{n})$ with probability 1*. In our extended network model, the average distance from a source node to a destination node is $\Theta(\sqrt{n})$ when they are uniformly distributed over S. Therefore, if nodes travel up to a distance $\Theta(\sqrt{n})$, for a certain time duration, the distance

[2] For simplicity, we omit the subscript π in λ_π unless confusion arises.

[3] Equivalently, the lower bound of flight length in the unit network model is generally assumed to be $1/\sqrt{n}$ [7].

between a source or a relay and a destination still remains $\Theta(\sqrt{n})$ on average which results in $O(1/\sqrt{n})$ throughput (see Lemma 1). Thus, it is obvious that a network aiming at obtaining $\omega(1/\sqrt{n})$ throughput must allow a delay which is no less than the maximum time duration that the first exit of a node from a disc of a radius $\Theta(\sqrt{n})$ does not occur with probability 1. This insight is formally described as follows:

$$C_{\Pi} = \sup\left\{t_n : \lim_{n\to\infty} \mathrm{P}\{T(r_n) > t_n\} = 1, r_n \in \Theta(\sqrt{n})\right\},$$

where $T(l)$ denotes the first exit time for a disc of a radius l and is defined as:

Definition 4 (First exit time): Let $\vec{X}^i(0) = x$.

$$T(l) \triangleq \inf\{t \geq 0 : \vec{X}^i(t) \notin B(x,l)\},$$

where $\vec{X}^i(t)$ denotes the location of node i at time t and $B(x,l)$ denotes the set of points y in S such that $d(x,y) \leq l$.

Lemma 1 ([1], [6]): Suppose that on average each packet is relayed over a total distance no less than $\Theta(\sqrt{n})$ in an extended network model. Then $\lambda = O(1/\sqrt{n})$.

VI. Critical Delay Analysis for Lévy Mobility

In this section, we provide detailed analysis to obtain critical delay for Lévy flight and Lévy walk. The first exit time analysis has been intensively studied in physics and mathematics. Specifically, trapping phenomenon (of a diffusing particle) in physics and its related theories have a direct connection to our first exit time problem. Motivated by this, we regard a mobile node as a diffusing particle in a finite interval $[0,\, 2l]$ with absorbing boundaries. Specifically the particle is assumed to be positioned at $x_0 = l$ at time $t = 0$, and eventually the particle is absorbed at either one of the end points. Then, the first exit time is the time taken to reach either of the absorbing boundaries.

Technical Approach. [6] obtains the first exit time distribution from the joint spatio-temporal PDF of a node (called occupation probability), which is well known and its closed-form solution is available [25]. However, solving the occupation probability of Lévy walk is very challenging. Instead of solving the occupation probability of Lévy walk, we decompose the occupation probability of BM to find the components constituting the occupation probability of BM and from this decomposition process, we identify the dominating terms influencing the first exit time. This is possible because the closed-form expression of BM's occupation probability is available. Our key observation is that the occupation probabilities and first exit time distributions of BM and Lévy model have similar structures in terms of dominating terms. Fortunately, finding the expression for the dominating terms for Lévy model is technically tractable. This allows us to study the limiting behavior of those dominating terms for Lévy model from which we can obtain the critical delay of Lévy model.

A. Brownian Motion

In this section, we elaborate on how the critical delay of BM can be obtained using the following three steps. (i) The occupation probability is obtained from the solution of a governing (differential or integral) equation. (ii) From this probability, we obtain the survival probability (which will be defined later), which in turn yields the first exit time distribution. (iii) By investigating the limiting behavior of the first exit time distribution, we can finally obtain the order of critical delay.

Step 1: We first project each node's position onto x-axis and y-axis. We then define for the projected processes $\{X^x_\alpha(t)\}_{t\geq 0}$ and $\{X^y_\alpha(t)\}_{t\geq 0}$ the first exit time similarly to that in Definition 4:

$$\begin{aligned} T_\alpha(l) &\triangleq \inf\{t \geq 0 : d(\vec{X}_\alpha(t), \vec{X}_\alpha(0)) \geq l\}, \\ T^x_\alpha(l) &\triangleq \inf\left\{t \geq 0 : |X^x_\alpha(t) - X^x_\alpha(0)| \geq l\right\}, \\ T^y_\alpha(l) &\triangleq \inf\left\{t \geq 0 : |X^y_\alpha(t) - X^y_\alpha(0)| \geq l\right\}. \end{aligned} \tag{1}$$

Random variables $T^x_\alpha(l)$ and $T^y_\alpha(l)$ represent the minimum time taken to exit a distance l from the initial position. Since the event $\{|X^x_\alpha(t) - X^x_\alpha(0)| \geq l\}$ implies the event $\{d(\vec{X}_\alpha(t), \vec{X}_\alpha(0)) \geq l\}$, we obtain

$$\mathrm{P}\{T^x_\alpha(l) \leq t\} \leq \mathrm{P}\{T_\alpha(l) \leq t\}. \tag{2}$$

In addition, by the union bound and the symmetry of Lévy mobility, we also obtain

$$\begin{aligned} \mathrm{P}\left\{T_\alpha(l) \leq t\right\} &\leq \mathrm{P}\{T^x_\alpha(l/\sqrt{2}) \leq t\} + \mathrm{P}\{T^y_\alpha(l/\sqrt{2}) \leq t\} \\ &= 2\mathrm{P}\{T^x_\alpha(l/\sqrt{2}) \leq t\}. \end{aligned} \tag{3}$$

By substituting $l = r_n \in \Theta(\sqrt{n})$ into (2) and (3) and combining (2) and (3), we have for all $t \geq 0$,

$$\mathrm{P}\{T^x_\alpha(r_n) \leq t\} \leq \mathrm{P}\{T_\alpha(r_n) \leq t\} \leq 2\mathrm{P}\{T^x_\alpha(r_n/\sqrt{2}) \leq t\}. \tag{4}$$

Note that the above inequality holds for all $\alpha \in (0, 2]$. Henceforth, we focus on the 1-D projected version of a 2-D BM, which is also a BM [6].

Let $P(x,t)$ denote the joint spatio-temporal PDF at position $x\,(\in [0,\, 2l])$ and time $t\,(\geq 0)$. We call $P(x,t)$ the *occupation probability*. Then, $P(x,t)$ is described by the following governing equation:

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{5}$$

where $D\,(> 0)$ is a diffusion coefficient. For a finite system $x \in [0,\, 2l]$ with absorbing boundaries, Equation (5) is subject to the boundary conditions $P(0,t) = P(2l,t) = 0 \;\forall t \geq 0$. Then, the solution of (5) is given by [25]:

$$P(x,t) = \sum_{i=1}^{\infty} A_i \sin\left(\frac{i\pi x}{2l}\right) \exp\left(-\left(\frac{i\pi}{2l}\right)^2 Dt\right),$$

where $A_i\,(i = 1, 2, \ldots)$ are determined from the initial condition $P(x, t=0) = \delta_{x,x_0}$ [4] and are given by $A_i = \frac{1}{l}\sin\left(\frac{i\pi}{2l}x_0\right)$. In our case of $x_0 \triangleq l$, we have

$$P(x,t) = \frac{1}{l}\sum_{i=1}^{\infty} \sin\left(\frac{i\pi}{2}\right) \sin\left(\frac{i\pi x}{2l}\right) \exp\left(-\left(\frac{i\pi}{2l}\right)^2 Dt\right).$$

Step 2: Let $S(t)$ be the probability that a node has not hit any absorbing boundary by time t. We call $S(t)$ the *survival probability*. Then, the survival probability can be obtained from the occupation probability $P(x,t)$ by $S(t) = \int_0^{2l} P(x,t)dx$ in general, and is given in the case of BM by

$$S(t) = \frac{2}{\pi}\sum_{i=1}^{\infty} \frac{1-\cos(i\pi)}{i} \sin\left(\frac{i\pi}{2}\right) \exp\left(-\left(\frac{i\pi}{2l}\right)^2 Dt\right).$$

[4] $\delta_{i,j}$ denotes the Kronecker delta, which is 1 if $i = j$ and 0 otherwise.

Note that the first exit time distribution can be obtained from the survival probability through the relation $\mathrm{P}\{T_\alpha^x(l) \le t\} = 1 - S(t)$ in general. Hence, in the case of BM (i.e., $\alpha = 2$), the first exit time distribution can be expressed as an infinite series of exponential functions as follows:

$$\mathrm{P}\{T_2^x(l) \le t\} = 1 - \sum_{i=1}^{\infty} \beta_i \exp\left(-\frac{\rho_i}{4l^2}t\right), \tag{6}$$

where $\beta_i \triangleq \frac{2\{1-\cos(i\pi)\}}{i\pi} \sin\left(\frac{i\pi}{2}\right)$ and $\rho_i \triangleq (i\pi)^2 D$.

Step 3: We are now ready to derive the main result of this subsection. By using the closed-form expression for $P\{T_2^x(l) \le t\}$ in (6), we can investigate the order of critical delay, stated in Lemmas 2 and 3.

Lemma 2 (Upper bound for BM): Suppose that time t in $\mathrm{P}\{T_2(r_n) \le t\}$ scales as $t \triangleq \hat{t}_n \in \Theta(n^{1+\epsilon})$ for some $\epsilon > 0$. Then, there does not exist any function $\mathcal{E}(n)$ such that $\lim_{n\to\infty} \mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_2(r_n) \le \hat{t}_n\} \le \mathcal{E}(n)$$

for $r_n \in \Theta(\sqrt{n})$.

Proof: We will prove this lemma by showing that $\lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n) \le \hat{t}_n\} = 1$. Then, from (4), we obtain

$$1 = \lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n) \le \hat{t}_n\} \le \lim_{n\to\infty} \mathrm{P}\{T_2(r_n) \le \hat{t}_n\},$$

i.e., $\lim_{n\to\infty} \mathrm{P}\{T_2(r_n) \le \hat{t}_n\} = 1$. Therefore, by contradiction, no such functions $\mathcal{E}(n)$ with $\lim_{n\to\infty} \mathcal{E}(n) = 0$ exist.

We substitute $l = r_n \in \Theta(\sqrt{n})$ into (6). Then, the series on the right-hand side of (6) when $t = \hat{t}_n$ becomes a function of n, and (for notational convenience) we let

$$\hat{S}(n) \triangleq \sum_{i=1}^{\infty} \beta_i \exp\left(-\frac{\rho_i}{4r_n^2}\hat{t}_n\right) = \sum_{i=1}^{\infty} \beta_i \exp\left(-\hat{c}\rho_i n^{\epsilon}\right) \tag{7}$$

for some constant $\hat{c} > 0$. We now need to take a limit to $\hat{S}(n)$. Note that when taking a limit to a function in the form of an infinite series, we need to interchange the order of limit and summation. To validate this interchange, we will show that the infinite series $\sum_{i=1}^{\infty} \beta_i \exp\left(-\hat{c}\rho_i n^{\epsilon}\right)$ converges uniformly on $\mathcal{D} \triangleq [1, \infty)$ by using the well-known Weierstrass M test [26].

The ith function $\beta_i \exp\left(-\hat{c}\rho_i n^{\epsilon}\right)$ in (7) is bounded by a constant $M_i \triangleq \frac{4}{\pi}\{\exp(-\hat{c}\pi^2 D)\}^i$ for all $n \in \mathcal{D}$ as follows:

$$|\beta_i \exp\left(-\hat{c}\rho_i n^{\epsilon}\right)| \le \frac{4}{\pi}\exp(-\hat{c}\rho_i) \le \frac{4}{\pi}\exp(-\hat{c}i\pi^2 D) = M_i,$$

where the first inequality comes from the bounds that $|\beta_i| \le \frac{4}{\pi}\ \forall i$ and $\exp(-\hat{c}\rho_i n^{\epsilon}) \le \exp(-\hat{c}\rho_i)\ \forall i$. In addition, the series $\sum_{i=1}^{\infty} M_i$ converges since it is a geometric series with a common ratio $\exp(-\hat{c}\pi^2 D) \in (0, 1)$. Since the target of the functions is a complete normed vector space, the infinite series $\sum_{i=1}^{\infty} \beta_i \exp\left(-\hat{c}\rho_i n^{\epsilon}\right)$ converges uniformly on $\mathcal{D}$ and consequently is continuous on $\mathcal{D}$.

Therefore, due to continuity on $\mathcal{D}$, we can interchange the order of limit and summation, and we finally have

$$\lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n) \le \hat{t}_n\} = 1 - \lim_{n\to\infty} \hat{S}(n)$$
$$= 1 - \sum_{i=1}^{\infty} \beta_i \lim_{n\to\infty} \exp\left(-\hat{c}\rho_i n^{\epsilon}\right) = 1,$$

which completes the proof. □

Lemma 3 (Lower bound for BM): Suppose that time t in $\mathrm{P}\{T_2(r_n) \le t\}$ scales as $t \triangleq \tilde{t}_n \in \Theta(n^{1-\epsilon})$ for some $\epsilon > 0$. Then, there exists a function $\mathcal{E}(n)$ such that $\lim_{n\to\infty} \mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_2(r_n) \le \tilde{t}_n\} \le \mathcal{E}(n)$$

for $r_n \in \Theta(\sqrt{n})$.

Proof: We will prove this lemma by showing that $\lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le \tilde{t}_n\} = 0$. Then, from (4), we obtain

$$\lim_{n\to\infty} \mathrm{P}\{T_2(r_n) \le \tilde{t}_n\} \le 2 \lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le \tilde{t}_n\} = 0,$$

i.e., $\lim_{n\to\infty} \mathrm{P}\{T_2(r_n) \le \tilde{t}_n\} = 0$, which is equivalent to showing the existence of a function $\mathcal{E}(n)$ such that $\mathrm{P}\{T_2(r_n) \le \tilde{t}_n\} \le \mathcal{E}(n)$ and $\lim_{n\to\infty} \mathcal{E}(n) = 0$.

We substitute $l = r_n/\sqrt{2} \in \Theta(\sqrt{n})$ into (6). Then, the series on the right-hand side of (6) when $t = \tilde{t}_n$ becomes a function of n, and analogous to the proof of Lemma 2, we let

$$\tilde{S}(n) \triangleq \sum_{i=1}^{\infty} \beta_i \exp\left(-\frac{\rho_i}{2r_n^2}\tilde{t}_n\right) = \sum_{i=1}^{\infty} \beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right) \tag{8}$$

for some constant $\tilde{c} > 0$. Similarly to the proof of Lemma 2, we will show that the infinite series $\sum_{i=1}^{\infty} \beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right)$ is continuous on $\mathcal{D} = [1, \infty)$.

For technical purposes, we restrict the domain of n as $\mathcal{D}_d \triangleq [1, d]$ for an arbitrary $d \ge 1$. Then, for all $n \in \mathcal{D}_d$, the ith function $\beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right)$ in (8) is bounded by a constant $N_i \triangleq \frac{4}{\pi}\{\exp(-\tilde{c}\pi^2 D d^{-\epsilon})\}^i$ as follows:

$$|\beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right)| \le \frac{4}{\pi}\exp(-\tilde{c}\rho_i d^{-\epsilon}) \le N_i.$$

In addition, the series $\sum_{i=1}^{\infty} N_i$ converges since it is a geometric series with a common ratio $\exp(-\tilde{c}\pi^2 D d^{-\epsilon}) \in (0, 1)$. Hence, the infinite series $\sum_{i=1}^{\infty} \beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right)$ converges uniformly on $\mathcal{D}_d$ and consequently is continuous on $\mathcal{D}_d$. Since d was arbitrary, we get continuity on $\mathcal{D}$.

Due to continuity on $\mathcal{D}$, we can interchange the order of limit and summation, and we have

$$\lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le \tilde{t}_n\} = 1 - \lim_{n\to\infty} \sum_{i=1}^{\infty} \beta_i \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right)$$
$$= 1 - \sum_{i=1}^{\infty} \beta_i \lim_{n\to\infty} \exp\left(-\tilde{c}\rho_i n^{-\epsilon}\right) = 1 - \sum_{i=1}^{\infty} \beta_i.$$

Note from (6) that $\mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le 0\} = 1 - \sum_{i=1}^{\infty} \beta_i$. In addition, it is clear that $\mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le 0\} = 0$. Hence, we have $\lim_{n\to\infty} \mathrm{P}\{T_2^x(r_n/\sqrt{2}) \le \tilde{t}_n\} = 0$. This completes the proof. □

By combining Lemmas 2 and 3, we can easily obtain the following theorem.

Theorem 1: **The critical delay under BM scales as $\Theta(n)$.**

Two remarks are in order.

Remark 1:

- The main idea behind the proof of Lemmas 2 and 3 was that the smallest (i.e., dominant) decay constant in the exponential functions (i.e., $\frac{\rho_1}{4l^2}$ in (6)) determines the limiting behavior of the first exit time distribution. That is, the smallest decay constant characterizes the critical delay for BM.

- Later we show that the first exit time distributions for Lévy mobility models have similar forms as that of BM. In the case of Lévy flight, the first exit time distribution can be expressed as an infinite series of exponential functions. In the case of Lévy walk, it can be expressed as a sum of an exponential function and a remaining term that decays faster than the exponential function. The mathematical technique in this subsection is used to prove that the dominant decay constant determines the order of critical delay for Lévy mobility models.

B. *Lévy Flight*

We follow the three steps used in analyzing BM.

Step 1: We first project each node's position onto x-axis and y-axis. Then, the projected processes satisfy the following property:

Lemma 4: For a given 2-D isotropic[5] Lévy flight $\{\vec{X}_\alpha(t)\}_{t\geq 0}$ of parameter α, its 1-D projection processes onto x-axis and y-axis (i.e., $\{X^x_\alpha(t)\}_{t\geq 0}$ and $\{X^y_\alpha(t)\}_{t\geq 0}$) are also isotropic Lévy flights of parameter α.

Proof: Note that $X^x_\alpha(t) = \sum_{i=1}^t Z_i \cos\theta_i$ and $X^y_\alpha(t) = \sum_{i=1}^t Z_i \sin\theta_i$, where Z_i and θ_i denote random variables representing the ith flight length and direction, respectively. Hence, it suffices to show that an arbitrary flight length of the projected processes (i.e., $|Z_i\cos\theta_i|$ and $|Z_i\sin\theta_i|$) follows a power-law type distribution with exponent α. By conditioning on the values of θ_i, we have the cumulative distribution function for $|Z_i\cos\theta_i|$ as

$$\begin{aligned}\mathrm{P}\{|Z_i\cos\theta_i| \leq x\} &= \int_0^{2\pi} \mathrm{P}\{|Z_i\cos\theta_i| \leq x \,|\, \theta_i = y\}\, dF_{\theta_i}(y) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{P}\{|Z_i\cos y| \leq x\}\, dy, \qquad (9)\end{aligned}$$

where the last equality follows from the independence of the flight length Z_i and direction θ_i. Since $\mathrm{P}\{|Z_i\cos y| \leq x\} \propto \frac{|\cos y|^\alpha}{x^\alpha}$, we have from (9) that

$$\mathrm{P}\{|Z_i\cos\theta_i| \leq x\} \propto \frac{1}{2\pi x^\alpha}\int_0^{2\pi} |\cos y|^\alpha dy \propto \frac{1}{x^\alpha},$$

which shows that the process $\{X^x_\alpha(t)\}_{t\geq 0}$ is Lévy flight. A noticeable result is that the 1-D projected version conserves the same mobility parameter α as that of the 2-D version.

We can similarly prove $\mathrm{P}\{|Z_i\sin\theta_i| \leq x\} \propto \frac{1}{x^\alpha}$, and thus the process $\{X^y_\alpha(t)\}_{t\geq 0}$ is also Lévy flight of parameter α. □

Recall that Equation (4) holds for all Lévy mobility. Hence, based on Lemma 4 and Equation (4), we focus on the 1-D Lévy flight in a finite interval $[0, 2l]$ with absorbing boundaries.

In contrast to BM, Lévy flight has the infinite second moment. As a consequence, Lévy flight can be described by using the fractional calculus [23] where the second order spatial derivative in (5) is replaced by the fractional derivative of order α with $0 < \alpha < 2$. That is, with continuous limit approximation [24], the occupation probability $P(x,t)$ is governed by the following fractional Fokker-Planck equation [27]:

$$\frac{\partial P(x,t)}{\partial t} = D_\alpha \frac{\partial^\alpha P(x,t)}{\partial x^\alpha}, \qquad (10)$$

where $D_\alpha\ (>0)$ is a scale factor. Analogous to the solution of (5), the solution of (10) can be expressed as

$$P(x,t) = \sum_{i=1}^\infty A_{\alpha,i}\psi_{\alpha,i}(x)\exp\left(-D_\alpha|\lambda_{\alpha,i}|t\right), \qquad (11)$$

where $A_{\alpha,i}\ (i = 1, 2, \ldots)$ are determined from the initial condition $P(x, t = 0) = \delta_{x,x_0}$ and are given by $A_{\alpha,i} = \psi_{\alpha,i}(x_0)$. The functions $\psi_{\alpha,i}(x)$ and the constants $\lambda_{\alpha,i}$ are solutions of the problem $\mathfrak{D}_\alpha[\psi_{\alpha,i}(x)] = \lambda_{\alpha,i}\psi_{\alpha,i}(x)$ for the operator $\mathfrak{D}_\alpha \triangleq \frac{\partial^\alpha}{\partial x^\alpha}$, and are called eigenfunctions and eigenvalues of $\mathfrak{D}_\alpha$, respectively. Without loss of generality, we assume that $\lambda_{\alpha,i}$ are arranged as $|\lambda_{\alpha,1}| < |\lambda_{\alpha,2}| < \ldots$.

In [27], Gitterman provided a solution of (10) which is widely accepted in physics, e.g., [28]. The eigenfunctions are given by $\psi_{\alpha,i}(x) = \sqrt{\frac{1}{l}}\sin\left(\frac{i\pi x}{2l}\right)$ and the eigenvalues are given by $\lambda_{\alpha,i} = -\left(\frac{i\pi}{2l}\right)^\alpha$. Here, the smallest (i.e., dominant) decay constant $|\lambda_{\alpha,1}|$ scales as $\Theta(l^{-\alpha})$. In [24], it is shown that the average first exit time for Lévy flight of parameter α with initial position $x_0 = l$ scales as $\Theta(l^\alpha)$, which induces that the dominant decay constant $|\lambda_{\alpha,1}|$ scales as $\Theta(l^{-\alpha})$.

Step 2: Similarly as done in Section VI-A, we can obtain the first exit time distribution $\mathrm{P}\{T^x_\alpha(l) \leq t\}$ by exploiting its relation with the occupation probability $P(x,t)$ and the survival probability $S(t)$ as follows:

$$\begin{aligned}\mathrm{P}\{T^x_\alpha(l) \leq t\} &= 1 - S(t) = 1 - \int_0^{2l} P(x,t)dx \\ &= 1 - \sum_{i=1}^\infty \beta_{\alpha,i}\exp\left(-D_\alpha|\lambda_{\alpha,i}|t\right), \qquad (12)\end{aligned}$$

where $\beta_{\alpha,i} \triangleq \psi_{\alpha,i}(x_0)\int_0^{2l}\psi_{\alpha,i}(x)dx = \frac{2}{\pi}\frac{1-\cos(i\pi)}{i}\sin(\frac{i\pi}{2})$.

Step 3: We now derive the main result of this subsection. By using the expression for $\mathrm{P}\{T^x_\alpha(l) \leq t\}$ in (12), we can investigate the order of critical delay, which is stated through the following two subsequent lemmas.

Lemma 5 (Upper bound for LF): Suppose that time t in $\mathrm{P}\{T_\alpha(r_n) \leq t\}$ scales as $t \triangleq \hat{t}_{\alpha,n} \in \Theta(n^{\frac{\alpha}{2}+\epsilon})$ for some $\epsilon > 0$. Then, there does not exist any function $\mathcal{E}(n)$ such that $\lim_{n\to\infty}\mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_\alpha(r_n) \leq \hat{t}_{\alpha,n}\} \leq \mathcal{E}(n)$$

for $r_n \in \Theta(\sqrt{n})$.

Proof: Since the dominant decay constant $|\lambda_{\alpha,1}|$ scales as $\Theta(l^{-\alpha}) = \Theta(r_n^{-\alpha}) = \Theta(n^{-\frac{\alpha}{2}})$, by using approaches in the proof of Lemma 2, we can show that $\lim_{n\to\infty}\mathrm{P}\{T_\alpha(r_n) \leq \hat{t}_{\alpha,n}\} = 1$. Therefore, by contradiction, no such functions $\mathcal{E}(n)$ with $\lim_{n\to\infty}\mathcal{E}(n) = 0$ exist. Due to similarities with the proof of Lemma 2, we omit detailed derivations. □

Lemma 6 (Lower bound for LF): Suppose that time t in $\mathrm{P}\{T_\alpha(r_n) \leq t\}$ scales as $t \triangleq \tilde{t}_{\alpha,n} \in \Theta(n^{\frac{\alpha}{2}-\epsilon})$ for some $\epsilon > 0$. Then, there exists a function $\mathcal{E}(n)$ such that $\lim_{n\to\infty}\mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_\alpha(r_n) \leq \tilde{t}_{\alpha,n}\} \leq \mathcal{E}(n)$$

[5] In general, an isotropic 2-D random walk refers to the walk that chooses its direction uniformly over $[0, 2\pi]$ at the beginning of each flight.

for $r_n \in \Theta(\sqrt{n})$.

Proof: Since the dominant decay constant $|\lambda_{\alpha,1}|$ scales as $\Theta(l^{-\alpha}) = \Theta(n^{-\frac{\alpha}{2}})$, by using approaches in the proof of Lemma 3, we can show that $\lim_{n\to\infty} \mathrm{P}\{T_\alpha(r_n) \le \tilde{t}_{\alpha,n}\} = 0$. This is equivalent to the existence of a function $\mathcal{E}(n)$ such that $\mathrm{P}\{T_\alpha(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathcal{E}(n)$ and $\lim_{n\to\infty} \mathcal{E}(n) = 0$. Due to similarities with the proof of Lemma 3, we omit detailed derivations. □

By combining Lemmas 5 and 6, we can easily obtain the following theorem.

Theorem 2: **The critical delay under Lévy flight of parameter α scales as $\Theta(n^{\frac{\alpha}{2}})$.**

In order to validate our derivations, we follow the technique in [7] and derive the order of critical delay for the unit network model. To apply the technique, we need to approximate Lévy flight by truncating its flight length to the range $[\frac{1}{\sqrt{n}}, 1]$. Lemma 7 summarizes the result for this approximated Lévy flight model.

Lemma 7: The critical delay under approximated Lévy flight of parameter α scales as $\Omega(n^{\frac{\alpha}{2}})$ for the unit network model.

Proof: Due to the limited space, the proof is given in [29]. □

The above result in Lemma 7 is identical to the result in Theorem 2, which partially verifies the validity of our derivation.

C. Lévy Walk

We begin with the description of differences between Lévy flight and Lévy walk from the perspective of mathematical modeling. It is clear that Lévy flight is of a class of discrete-time Markov processes. In contrast to Lévy flight, the spatiotemporal coupling of Lévy walk makes the future motion of a Lévy walker dependent on its past trajectory. On the other hand, at each turning point, the position of the next turning point is chosen independently of the past trajectory. Thus, Lévy walk is known to be of a class of semi-Markov processes [22], and consequently it induces the following two technical difficulties: (i) The technique by Sharma *et al*. in [7] cannot be applicable because it requires decoupling of space and time. (ii) The governing equation for the occupation probability $P(x,t)$ should be described by integral equations rather than differential equations used in BM and Lévy flight [22].

We again follow the three steps.

Step 1: The argument in the proof of Lemma 4 also shows that, for a given 2-D isotropic Lévy walk, its 1-D projected versions are also isotropic processes with the same flight length distribution as that of the 2-D Lévy walk. However, the velocity of the projected processes is not a constant for every flight. Therefore, in contrast to Lévy flight, the 1-D projected versions of the 2-D Lévy walk are not 1-D Lévy walks. In the following lemma, we derive a relationship between the first exit times for 2-D Lévy walk (i.e., $T_\alpha(r_n)$), 1-D projected process (i.e., $T^x_\alpha(r_n)$) and 1-D Lévy walk (i.e., $T^{\mathrm{1D}}_\alpha(r_n)$).

Lemma 8: Fix $\alpha \in (0,2)$. Then, for any $\eta \in (0,1)$ there exists $\delta = \delta(\eta) \in (0,1)$ such that $\mathrm{P}\left\{T^{\mathrm{1D}}_\alpha(r_n) \le (t-2r_n)\delta\right\} - 1 + \eta \le \mathrm{P}\{T^x_\alpha(r_n) \le t\} \le \mathrm{P}\{T_\alpha(r_n) \le t\}$ holds $\forall n \in \mathbb{N}$.

Proof: Due to the limited space, the proof is given in [29]. □

By virtue of Lemma 8, we henceforth focus on a 1-D Lévy walk in a finite interval $[0,\, 2l]$ with absorbing boundaries. Let $Q(x,t)$ denote the probability that the Lévy walker changes its direction at location x at time t. We call $Q(x,t)$ the *turning point distribution*. We show later that the turning point distribution essentially determines the behavior of the occupation probability. By conditioning on $Q(x,t)$, the occupation probability $P(x,t)$ can be expressed as follows [22]:

$$P(x,t) = \frac{1}{2}\int_0^{2l}\int_0^t Q(x',t') \times \mathrm{P}\{Z_\alpha \ge |x-x'|\}\delta(|x-x'| - v(t-t'))dt'dx',$$

where v is a constant velocity of a Lévy walker and is normalized to $v \triangleq 1$ without loss of generality. In [22], the Laplace transform to the temporal domain was used to solve the above integral equation, and it was shown that

$$\hat{P}(x,s) = \frac{1}{2}\int_0^{2l} \hat{Q}(x',s) \times \mathrm{P}\{Z_\alpha \ge |x-x'|\}\exp(-s|x-x'|)dx', \quad (13)$$

where $\hat{P}(x,s) \triangleq \int_0^\infty \exp(-st)P(x,t)dt$ and $\hat{Q}(x,s) \triangleq \int_0^\infty \exp(-st)Q(x,t)dt$ are the Laplace transforms of $P(x,t)$ and $Q(x,t)$ to the temporal domain, respectively.

Analogous to the solution (11), $\hat{Q}(x,s)$ can be expressed in terms of eigenfunctions and eigenvalues as

$$\hat{Q}(x,s) = \sum_{i=1}^{\infty} B_{\alpha,i}(s)\frac{\phi_{\alpha,i}(x,s)}{\{\xi_{\alpha,i}(s)\}^{-1} - 1},$$

where $B_{\alpha,i}(s)\,(i = 1,2,\ldots)$ are determined from the initial condition $Q(x,t=0) = \delta_{x,x_0}$ and are given by $B_{\alpha,i}(s) = \phi_{\alpha,i}(x_0,s)$. The functions $\phi_{\alpha,i}(x,s)$ and the constants $\xi_{\alpha,i}(s)$ are solutions of the problem $\mathfrak{F}_\alpha[\phi_{\alpha,i}(x,s)] = \xi_{\alpha,i}(s)\phi_{\alpha,i}(x,s)$ for the operator $\mathfrak{F}_\alpha[\phi(x,s)] \triangleq \int_0^{2l}\exp(-s|x-x'|)f_{Z,\alpha}(x-x')\phi(x',s)dx'$, and are called eigenfunctions and eigenvalues of $\mathfrak{F}_\alpha$, respectively.

To the best of our knowledge, closed-form formulas for $\phi_{\alpha,i}(x,s)$ and $\xi_{\alpha,i}(s)$ have not been explored to date. However, it was proved that $\hat{Q}(x,s)$ has a countably infinite set of simple negative poles[6] [22]. It can be seen from (13) that $\hat{P}(x,s)$ has the same poles as $\hat{Q}(x,s)$ since $\exp(-s|x-x'|)$ has no poles. This shows that the pole with the smallest absolute value, denoted by $\eta_{\alpha,1}\,(>0)$, determines the behavior of the occupation probability $P(x,t)$ for large t as follows:

$$P(x,t) \propto \exp(-\eta_{\alpha,1}t).$$

In [22], $\eta_{\alpha,1}$ is investigated and found to scale at large l as $\Theta(l^{-1})$ for $0 < \alpha < 1$ and $\Theta(l^{-\alpha})$ for $1 \le \alpha < 2$. In [24], it is shown that the average first exit time for Lévy walk of parameter α with initial position $x_0 = l$ scales as $\Theta(l)$ for $0 < \alpha < 1$ and $\Theta(l^\alpha)$ for $1 \le \alpha < 2$. This induces that the smallest (i.e., dominant) pole $\eta_{\alpha,1}$ scales as $\Theta(l^{-1})$ for $0 < \alpha < 1$ and $\Theta(l^{-\alpha})$ for $1 \le \alpha < 2$.

Step 2: Similarly as in Sections VI-A and VI-B, we can obtain the first exit time distribution $\mathrm{P}\{T^{\mathrm{1D}}_\alpha(l) \le t\}$ by exploiting its relation with the occupation probability $P(x,t)$

[6]For a rational function $f(s) = \frac{N(s)}{D(s)}$, a pole $s^\star$ is defined to be a value such that $D(s^\star) = 0$. A pole of order 1 is called a simple pole.

and the survival probability $S(t)$ as follows:

$$\begin{aligned}\mathrm{P}\{T_\alpha^{\mathrm{1D}}(l) \le t\} &= 1 - S(t) = 1 - \int_0^{2l} P(x,t)dx \\ &= 1 - c_\alpha \exp\left(-\eta_{\alpha,1} t\right) + R_\alpha(t), \end{aligned} \tag{14}$$

where c_α is a constant and $R_\alpha(t)$ denotes a remaining term that decays faster (to zero) than the function $\exp(-\eta_{\alpha,1}t)$.

Step 3: We now derive the main result of this subsection. By using the expression for $\mathrm{P}\{T_\alpha^{\mathrm{1D}}(l) \le t\}$ in (14) and Lemma 8, we can investigate the order of critical delay, which is stated through the following two subsequent lemmas.

Lemma 9 (Upper bound for LW): Suppose that, for some $\epsilon > 0$, time t in $\mathrm{P}\{T_\alpha(r_n) \le t\}$ scales as $t \triangleq \hat{t}_{\alpha,n} \in \Theta(n^{\frac{1}{2}+\epsilon})$ for $\alpha \in (0,1)$ and $t \triangleq \hat{t}_{\alpha,n} \in \Theta(n^{\frac{\alpha}{2}+\epsilon})$ for $\alpha \in [1,2)$. Then, there does not exist any function $\mathcal{E}(n)$ such that $\lim_{n\to\infty} \mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_\alpha(r_n) \le \hat{t}_{\alpha,n}\} \le \mathcal{E}(n)$$

for $r_n \in \Theta(\sqrt{n})$.

Proof: We will prove this lemma by showing that $\lim_{n\to\infty} \mathrm{P}\{T_\alpha^{\mathrm{1D}}(r_n) \le (\hat{t}_{\alpha,n} - r_n)\delta\} = 1$. Then, from Lemma 8, we obtain $\lim_{n\to\infty} \mathrm{P}\{T_\alpha(r_n) \le \hat{t}_{\alpha,n}\} \ge \eta > 0$. Therefore, by contradiction, no such functions $\mathcal{E}(n)$ with $\lim_{n\to\infty} \mathcal{E}(n) = 0$ exist.

We first consider the case of $0 < \alpha < 1$ and substitute $l = r_n \in \Theta(\sqrt{n})$ into (14). Since l scales as $\Theta(\sqrt{n})$, $\eta_{\alpha,1}$ scales as $\Theta(l^{-1}) = \Theta(n^{-\frac{1}{2}})$ in this case. Hence, the exponential function on the right-hand side of (14) when $t = \hat{t}_{\alpha,n}$ becomes $c_\alpha \exp(-\eta_{\alpha,1}\hat{t}_{\alpha,n}) = c_\alpha \exp(-\hat{c}_\alpha n^\epsilon)$ for some constant $\hat{c}_\alpha > 0$, and accordingly we have in the limit

$$\lim_{n\to\infty} c_\alpha \exp(-\eta_{\alpha,1}\hat{t}_{\alpha,n}) = \lim_{n\to\infty} c_\alpha \exp(-\hat{c}_\alpha n^\epsilon) = 0. \tag{15}$$

Since the remaining term $R_\alpha(t)$ approaches faster to zero than $\exp(-\eta_{\alpha,1}t)$, we have in the limit $\lim_{n\to\infty} |R_\alpha(\hat{t}_{\alpha,n})| \le \lim_{n\to\infty} \exp(-\eta_{\alpha,1}\hat{t}_{\alpha,n}) = 0$, from which we obtain

$$\lim_{n\to\infty} R_\alpha(\hat{t}_{\alpha,n}) = 0. \tag{16}$$

By combining (15) and (16), we obtain $\lim_{n\to\infty} \mathrm{P}\{T_\alpha^{\mathrm{1D}}(r_n) \le \hat{t}_{\alpha,n}\} = 1$ *for any* $\hat{t}_{\alpha,n} \in \Theta(n^{\frac{1}{2}+\epsilon})$. Note that we still have $(\hat{t}_{\alpha,n} - 2r_n)\delta \in \Theta(n^{\frac{1}{2}+\epsilon})$ since $\delta > 0$ is fixed as a constant (regardless of n) and $r_n \in \Theta(n^{\frac{1}{2}})$. Therefore we finally obtain $\lim_{n\to\infty} \mathrm{P}\{T_\alpha^{\mathrm{1D}}(r_n) \le (\hat{t}_{\alpha,n} - r_n)\delta\} = 1$.

In the case of $1 \le \alpha < 2$, we can prove similarly as above and omit detailed derivations. □

Lemma 10 (Lower bound for LW): Suppose that, for some $\epsilon > 0$, time t in $\mathrm{P}\{T_\alpha(r_n) \le t\}$ scales as $t \triangleq \tilde{t}_{\alpha,n} \in \Theta(n^{\frac{1}{2}-\epsilon})$ for $\alpha \in (0,1)$ and $t \triangleq \tilde{t}_{\alpha,n} \in \Theta(n^{\frac{\alpha}{2}-\epsilon})$ for $\alpha \in [1,2)$. Then, there exists a function $\mathcal{E}(n)$ such that $\lim_{n\to\infty} \mathcal{E}(n) = 0$ and

$$\mathrm{P}\{T_\alpha(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathcal{E}(n)$$

for $r_n \in \Theta(\sqrt{n})$.

Proof: We will prove this lemma by specifying functions $\mathcal{E}(n)$ that satisfy $\lim_{n\to\infty} \mathcal{E}(n) = 0$ and $\mathrm{P}\{T_\alpha(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathcal{E}(n)$ for each of the cases of $0 < \alpha < 1$ and $1 \le \alpha < 2$.

We first consider the case of $0 < \alpha < 1$. Since a Lévy walker moves with a constant velocity $v = 1$, it takes at least $r_n \in \Theta(\sqrt{n})$ time to exit from a disc of a radius r_n. Therefore, it is obvious that $\mathrm{P}\{T_\alpha(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathrm{P}\{T_\alpha(r_n) < r_n\} = 0$.

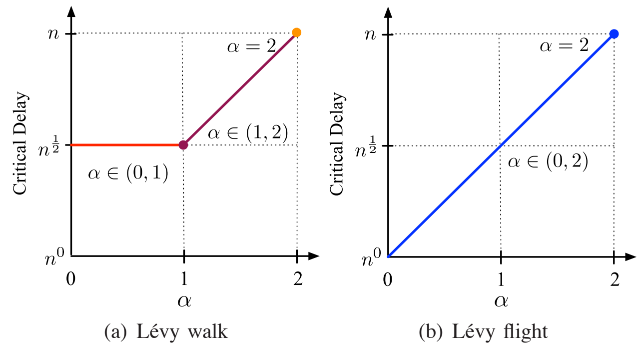

Fig. 2. Critical delays of Lévy walk and Lévy flight mobility models for different α.

By choosing $\mathcal{E}(n) \triangleq 0\ \forall n$, we have proved the lemma in the case of $0 < \alpha < 1$.

We next consider the case of $1 \le \alpha < 2$. Since Lévy flight takes a constant time for each flight whereas Lévy walk takes a constant velocity, we have with probability 1 that $T_{\alpha,\mathrm{LF}}(r_n) \le T_{\alpha,\mathrm{LW}}(r_n)$, where the subscripts LF and LW are added to the definition (1) to distinguish the first exit times between Lévy flight and Lévy walk. In addition, from Lemma 6, there exists a function $\mathcal{E}_{\mathrm{LF}}(n)$ such that $\lim_{n\to\infty} \mathcal{E}_{\mathrm{LF}}(n) = 0$ and $\mathrm{P}\{T_{\alpha,\mathrm{LF}}(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathcal{E}_{\mathrm{LF}}(n)$. We choose $\mathcal{E}(n) \triangleq \mathcal{E}_{\mathrm{LF}}(n)$ for $1 \le \alpha < 2$. Then, $\mathrm{P}\{T_{\alpha,\mathrm{LW}}(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathrm{P}\{T_{\alpha,\mathrm{LF}}(r_n) \le \tilde{t}_{\alpha,n}\} \le \mathcal{E}_{\mathrm{LF}}(n) = \mathcal{E}(n)$, where the first inequality comes from the property that $T_{\alpha,\mathrm{LF}}(r_n) \le T_{\alpha,\mathrm{LW}}(r_n)$. This completes the proof. □

By combining Lemmas 9 and 10, we can obtain the following theorem.

Theorem 3: **The critical delay under Lévy walk of parameter $\boldsymbol{\alpha}$ scales as $\boldsymbol{\Theta(n^{\frac{1}{2}})}$ for $\boldsymbol{0 < \alpha < 1}$ and $\boldsymbol{\Theta(n^{\frac{\alpha}{2}})}$ for $\boldsymbol{1 \le \alpha < 2}$.**

VII. Concluding Remarks

We summarize the high-level interpretations of this paper. Fig. 2 shows the critical delay of Lévy walk and Lévy flight, parameterized by α. Lévy flight shows that critical delay proportionally increases with α. However, in Lévy walk, we can find a phase transition that for $0 < \alpha < 1$, the critical delay is constantly $\Theta(n^{1/2})$ and shifts to the proportional increasing phase for $1 \le \alpha \le 2$. Two different scaling regions are essentially related to the fact that the mean flight length of Lévy walk for $0 < \alpha < 1$ is *infinite* but finite for $1 \le \alpha \le 2$. In contrast to Lévy walk, the travel time independence of flight length in Lévy flight leads to continuous scaling over α. Note that for $\alpha = 2$ (i.e., BM) our result coincides with that in [7] which also studied the critical delay of BM.

Our results can take α from experimental measurements from [8], to determine how the network delay with human mobility scales in practice. To give insight to the readers, we show α values measured from five different sites in Table I presented in [8] with a flight extraction method, "rectangle" [7]. We see that critical delays for human mobility range from

[7] We do not present α values from other extraction methods in [8] which intentionally exclude some detailed motions of real traces. To capture specific behaviors of humans, one can borrow those α values.

TABLE I
EXPERIMENTAL α VALUES FOR DIFFERENT SITES PRESENTED IN [8].

Site	α	Site	α
KAIST	0.53	New York City	1.62
NCSU	1.27	Disney World	1.20
		State fair	1.81

$\Theta(n^{0.27})$ to $\Theta(n^{0.91})$. Human mobility mainly have $\alpha > 1$, which necessitates longer delay than $\Theta(\sqrt{n})$, the average delay for static multi-hop networks with a constant packet size where the average delay is larger than critical delay by definition. This implies that it may be hard to design a low delay protocol for mobile networks under human mobility.

Our contribution is not restricted to the mathematical derivation of delay scaling for new mobility models. We also provided a technique that connects the diffusion equation of a continuous time random walk process to the delay scaling. This technique can be extended to the analysis of other detailed metrics, e.g., end-to-end delay distribution of flows.

Future work includes investigation of performance scaling for mobile networks with heterogeneous and collective node mobilities. In addition to the recent research topics on "per-node throughput scaling" under inhomogeneous spatial node distributions (i.e., Cox process, Neyman-Scott process, Matérn cluster process and Thomas process), e.g., [30], [31], our paper can be an important step to the study of delay scaling under such heterogeneous networks. There is an insight from [9] that in human-assisted networks, the actual delays might be even shorter. This is because people's mobility is not completely random: people tend to visit the same locations and meet a similar set of people every day. Although their mobility can be characterized by heavy-tail distributions, these regularity in daily mobility of people could make it much easier to route packets among people (as long as they are socially connected). Therefore, there remains a possibility of designing a low delay protocol for mobile networks under heterogeneous human mobility by judiciously utilizing these social factors.

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